1. How many three-digit positive integers have digits which sum to a multiple of 10?

   Answer: 90

   Solution: If the first digit is a and the second digit is b, then there will always be exactly one valid choice for the third digit, which is the digit equivalent to $-a-b \pmod{10}$. Since there are 9 possibilities for the first digit and 10 for the second digit, the answer is $9 \cdot 10 = \boxed{90}$.

2. A positive integer is called *extra-even* if all of its digits are even. Compute the number of positive integers n less than or equal to 2022 such that both n and $2n$ are both extra-even.

   Answer: 31

   Solution: We claim that a positive integer n has both n and $2n$ extra-even if and only if all of its digits are in $\{0, 2, 4\}$. Note that this is sufficient because n will have all of its digits in $\{0, 2, 4\}$ and $2n$ will have all of its digits in $\{0, 4, 8\}$.

   To see that this is necessary, we note that n must have all of its digits in $\{0, 2, 4, 6, 8\}$, so we just need to rule out 6s and 8s. If n has a 6 or 8, then we can write

   $$n = 10^{a+1}n_1 + 10^a d + n_2,$$

   where n_1 and n_2 are extra-even, and $d \in \{6, 8\}$. Then

   $$2n = 10^{a+1}(2n_1 + 1) + 10^a(2d - 10) + 2n_2,$$

   so we see that the digit in the 10^{a+1} place (which is the last digit of $2n_1 + 1$) must have an odd digit. Thus, $2n$ is not extra-even.

   We now count the number of positive integers less than or equal to 2022 with all digits in $\{0, 2, 4\}$. We have the following cases.

   - At most three digits: the number of nonnegative integers with at most three digits and digits in $\{0, 2, 4\}$ is $3 \cdot 3 \cdot 3$, but we must then exclude 000. So we have $3 \cdot 3 \cdot 3 - 1$ options here.
   - Four digits: we must begin with 20__. The next digit is either a 0, which gives three options for the last digit, or a 2, which gives two options for the last digit.

   Totaling the above, we have $3 \cdot 3 \cdot 3 - 1 + 3 + 2 = 27 - 1 + 5 = \boxed{31}$ total positive integers.

3. Let A be the product of all positive integers less than 1000 whose ones or hundreds digit is 7. Compute the remainder when $A/101$ is divided by 101.

   Answer: 19

   Solution: We will use Wilson's theorem to get rid of blocks of numbers.

   - Consider the block of numbers which have a ones digit of 7. We claim that the numbers 7, 17, 27, ..., 997, not including 707, cover all nonzero remainders except for $1007 \equiv -3 \pmod{101}$. This is because all of these numbers are of the form $7 + 10i$, for $i \in \{0, \dots, 99\}$. Note that if any $7 + 10i \equiv 7 + 10j \pmod{101}$, then $10(i - j) \equiv 0 \pmod{101}$, and since since $\gcd(10, 101) = 1$, this implies that $101 \mid i - j$. But this never happens for distinct $i, j \in \{0, \dots, 99\}$, so all of the numbers are distinct modulo 101. Aside from 707, there are 99 of them, so the claim follows. Therefore by Wilson's theorem,

   $$\frac{7 \cdot 17 \cdot 27 \cdots 997}{101} \equiv (-1) \cdot 7 \cdot (-3)^{-1} \equiv 7 \cdot 3^{-1} \pmod{101}.$$

- Consider the block of numbers which have a hundreds digit of 7. The numbers 700, 701, . . . , 799, not including 707, cover all nonzero remainders except for $800 \equiv -8 \pmod{101}$. Therefore by Wilson's theorem,

$$\frac{700 \cdot 701 \cdots 799}{101} \equiv (-1) \cdot 7 \cdot (-8)^{-1} \equiv 7 \cdot 8^{-1} \pmod{101}.$$

- Lastly, the above two blocks both include 707, 717, . . . , 797, so we need to divide out by a factor of 7 from the 707 and also divide out by

$$\begin{aligned} 717 \cdot 727 \cdots 797 &\equiv 10 \cdot 20 \cdots 90 \\ &\equiv 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10^9 \\ &\equiv 3 \cdot 4 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 100^5 \\ &\equiv 29 \cdot (-1)^5 \\ &\equiv 72 \pmod{101}. \end{aligned}$$

In total, we combine the above to compute

$$\begin{aligned} \frac{A}{101} &\equiv 7 \cdot 3^{-1} \cdot 7 \cdot 8^{-1} \cdot 7^{-1} \cdot 72^{-1} \\ &\equiv 7 \cdot (3 \cdot 8 \cdot 72)^{-1} \\ &\equiv 7 \cdot 11^{-1} \\ &\equiv 7 \cdot 46 \\ &\equiv \boxed{19} \pmod{101}. \end{aligned}$$