1. Marisela is putting on a juggling show! She starts with 1 ball, tossing it once per second. Lawrence tosses her another ball every five seconds, and she always tosses each ball that she has once per second. (Marisela tosses her first ball at the 1st second, and starts tossing the second ball at the 6th second. Tosses at the 60th second also count.) Compute the total number of tosses Marisela has made one minute after she starts juggling.

   Answer: 390

   Solution: In the first five seconds, Marisela has one ball, so she makes $5 \cdot 1 = 5$ tosses. In the next five seconds, she has two balls, making $5 \cdot 2 = 10$ tosses. The total number of tosses is

   $$5 \cdot 1 + 5 \cdot 2 + 5 \cdot 3 + \cdots + 5 \cdot 12 = 5 \cdot \frac{12 \cdot 13}{2} = \boxed{390}$$

   since there are $60 = 5 \cdot 12$ seconds in a minute.

2. Let a and b be the roots of the polynomial $x^2 + 2020x + c$. Given that $\frac{a}{b} + \frac{b}{a} = 98$, compute $\sqrt{c}$.

   Answer: 202

   Solution 1: *(Vieta's formulae.)* We have that $a + b = -2020$ and $ab = c$ by Vieta's formulae. This gives us

   $$\frac{a}{b} + \frac{b}{a} = \frac{a^2 + b^2}{ab} = \frac{a^2 + 2ab + b^2}{ab} - 2 = \frac{(a+b)^2}{ab} - 2 = \frac{2020^2}{c} - 2 = 98.$$

   Now we can solve for c to find

   $$c = \frac{2020^2}{100} \implies \sqrt{c} = \frac{2020}{10} = \boxed{202}$$

   which is our answer.

   Solution 2: *(Calculation of roots.)* We use the quadratic formula to find the roots

   $$a = \frac{-2020 + \sqrt{2020^2 - 4c}}{2} = -1010 + \sqrt{1010^2 - c}$$
   $$b = \frac{-2020 - \sqrt{2020^2 - 4c}}{2} = -1010 - \sqrt{1010^2 - c},$$

   where we chose the plus and minus signs arbitrarily. Now we have

   $$\frac{a}{b} + \frac{b}{a} = \frac{-1010 + \sqrt{1010^2 - c}}{-1010 - \sqrt{1010^2 - c}} + \frac{-1010 - \sqrt{1010^2 - c}}{-1010 + \sqrt{1010^2 - c}}.$$

   We rationalize both denominators by multiplying by conjugates to yield

   $$\begin{aligned}\frac{a}{b} + \frac{b}{a} &= \frac{(-1010 + \sqrt{1010^2 - c})^2}{c} + \frac{(-1010 - \sqrt{1010^2 - c})^2}{c} \\ &= \frac{1010^2 - 2020\sqrt{1010^2 - c} + 1010^2 - c + 1010^2 + 2020\sqrt{1010^2 - c} + 1010^2 - c}{c} \\ &= \frac{4 \cdot 1010^2 - 2c}{c} = \frac{2020^2}{c} - 2 = 98,\end{aligned}$$

   and now we can solve for c as in solution 1 to yield $\sqrt{c} = \boxed{202}$.

3. The graph of the degree 2021 polynomial $P(x)$, which has real coefficients and leading coefficient 1, meets the x-axis at the points $(1,0),(2,0),(3,0),\dots,(2020,0)$ and nowhere else. The mean of all possible values of $P(2021)$ can be written in the form $a!/b$, where a and b are positive integers and a is as small as possible. Compute $a+b$.

   Answer: 2023

   Solution: Since $P(x)$ has degree 2021 and has real roots $1,2,\dots,2020$, the 2021st root must also be real and hence be an element of the set $\{1,2,\dots,2020\}$. That is, one of the numbers $1,2,\cdots,2020$ is a double root. Let this double root be r. Then

$$P(x) = (x-1)(x-2)\cdots(x-2020)\cdot(x-r),$$

   so

$$P(2021) = 2020\cdot 2019\cdot 2018\cdot\dots\cdot 1\cdot(2021-r) = 2020!\cdot(2021-r).$$

   Because r is taken from $\{1,2,\cdots,2020\}$, the average of the possible values of $P(2021)$ is

$$\frac{1}{2020}\cdot\sum_{r=1}^{2020} 2020!\cdot(2021-r) = \frac{1}{2020}\cdot 2020!\cdot(2020+2019+\cdots+2+1) = \frac{1}{2020}\cdot 2020!\cdot\frac{2020\cdot 2021}{2} = \frac{2021!}{2}.$$

   Hence, we have $a=2021$ and $b=2$, and $a+b = 2021+2 = \boxed{2023}$.

4. Let φ be the positive solution to the equation

$$x^2 = x+1.$$

   For $n \geq 0$, let a_n be the unique integer such that $\varphi^n - a_n\varphi$ is also an integer. Compute

$$\sum_{n=0}^{10} a_n.$$

   Answer: 143

   Solution 1: The main tool for this problem is the fact that $\varphi^2 = \varphi + 1$. If we let $b_n = \varphi^n - a_n\varphi$, we have

$$\varphi^n = a_n\varphi + b_n \implies \varphi^{n+1} = a_n\varphi^2 + b_n\varphi = (a_n+b_n)\varphi + a_n.$$

   We can see now (with the additional note that $b_1 = a_0$) that $b_n = a_{n-1}$ for all n, so we have the recurrence $a_{n+1} = a_n + a_{n-1}$, meaning that the sequence (a_n) is actually the Fibonacci sequence with $a_0 = 0, a_1 = 1$. We can calculate the answer directly as

$$\sum_{n=0}^{10} a_n = 0+1+1+2+3+5+8+13+21+34+55 = \boxed{143}.$$

   Solution 2: We can look at the partial sums

$$0,1,2,4,7,12,20,33,\cdots$$

   and realize that they are one less than the Fibonacci numbers, which leads us to

$$\sum_{n=0}^{10} a_n = a_{12} - 1 = \boxed{143}$$

yet again. More generally, we can prove the identity

$$\sum_{k=0}^{n} a_k = a_{n+2} - 1$$

by observing

$$\sum_{k=0}^{n} \varphi^k = \left(\sum_{k=0}^{n} a_k\right)\varphi + \left(1 + \sum_{k=0}^{n-1} a_k\right)$$

and

$$\sum_{k=0}^{n} \varphi^k = \frac{\varphi^{n+1}-1}{\varphi - 1} = \frac{\varphi^{n+1}-1}{1/\varphi} = \varphi^{n+2} - \varphi,$$

whence

$$\varphi^{n+2} = \varphi + \sum_{k=0}^{n} \varphi^k = \left(1 + \sum_{k=0}^{n} a_k\right)\varphi + \left(1 + \sum_{k=0}^{n-1} a_k\right).$$

Now, because there is only one way to write φ^{n+2} as $a\varphi + b$ with $a, b \in \mathbb{Z}$ (this was given to us in the problem but also follows directly from the fact that φ is irrational), we must have that

$$a_{n+2} = 1 + \sum_{k=0}^{n} a_k \implies \sum_{k=0}^{n} a_k = a_{n+2} - 1$$

as desired.

5. Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a function such that for all $x, y \in \mathbb{R}^+$, $f(x)f(y) = f(xy) + f\left(\frac{x}{y}\right)$, where $\mathbb{R}^+$ represents the positive real numbers. Given that $f(2) = 3$, compute the last two digits of $f\left(2^{2^{2020}}\right)$.

Answer: 47

Solution: Observe that, setting $x = y$, we have

$$f(x)^2 = f(x^2) + f(1).$$

Also, setting $x = 2$ and $y = 1$ gives

$$f(2)f(1) = f(2) + f(2),$$

so $3f(1) = 6$ and $f(1) = 2$. It follows that $f(x)^2 = f(x^2) + 2$, so $f(x^2) = f(x)^2 - 2$. Using this recurrence, we find that

$$\begin{aligned}
f\left(2^{2^0}\right) &= 3 \equiv 3 \pmod{100} \\
f\left(2^{2^1}\right) &= 7 \equiv 7 \pmod{100} \\
f\left(2^{2^2}\right) &= 47 \equiv 47 \pmod{100} \\
f\left(2^{2^3}\right) &= 2207 \equiv 7 \pmod{100} \\
f\left(2^{2^4}\right) &\equiv 7^2 - 2 = 47 \pmod{100} \\
f\left(2^{2^5}\right) &\equiv 47^2 - 2 \equiv 7 \pmod{100} \\
&\vdots
\end{aligned}$$

For a even, $f\left(2^{2^a}\right) \equiv 47 \pmod{100}$. In particular, for $a = 2020$, we find that the last two digits of $f\left(2^{2^{2020}}\right)$ are $\boxed{47}$. For good measure, we show that a function f satisfying the criteria presented in the problem statement exists. We note that for $a, x, y > 0$, we have

$$\left(x^a + x^{-a}\right)\left(y^a + y^{-a}\right) = (xy)^a + \left(\frac{x}{y}\right)^a + \left(\frac{y}{x}\right)^a + \left(\frac{1}{xy}\right)^a = (xy)^a + (xy)^{-a} + \left(\frac{x}{y}\right)^a + \left(\frac{x}{y}\right)^{-a},$$

so the function

$$f(x) = x^a + x^{-a}$$

satisfies the given functional equation whenever a is positive (a could be negative as well, but since a and $-a$ yield the same function f, we can assume just as well that $a > 0$). Now we need only solve for a such that $f(2) = 3$. This gives

$$3 = 2^a + 2^{-a} \implies (2^a)^2 - 3(2^a) + 1 = 0 \implies 2^a = \frac{3 \pm \sqrt{5}}{2}.$$

Only one of these roots makes $a > 0$, so we have

$$a = \log_2 \frac{3 + \sqrt{5}}{2} = 2\log_2 \varphi,$$

where φ is the golden ratio. Hence, the function

$$f(x) = x^{2\log_2 \varphi} + x^{-2\log_2 \varphi} = \varphi^{2\log_2 x} + \varphi^{-2\log_2 x}$$

satisfies all of the given conditions. This suggests an interesting relationship between the given function and the Fibonacci numbers that enthusiastic contestants are urged to pursue.

6. Given that $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, the value of

$$\sum_{n=3}^{10} \frac{\binom{n}{2}}{\binom{n}{3}\binom{n+1}{3}}$$

can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.

Answer: 329

Solution: Notice using Pascal's identity that

$$\frac{\binom{n}{2}}{\binom{n}{3}\binom{n+1}{3}} = \frac{\binom{n+1}{3} - \binom{n}{3}}{\binom{n}{3}\binom{n+1}{3}} = \frac{1}{\binom{n}{3}} - \frac{1}{\binom{n+1}{3}},$$

so

$$\begin{aligned}\sum_{n=3}^{10} \frac{\binom{n}{2}}{\binom{n}{3}\binom{n+1}{3}} &= \left(\frac{1}{\binom{3}{3}} - \frac{1}{\binom{4}{3}}\right) + \left(\frac{1}{\binom{4}{3}} - \frac{1}{\binom{5}{3}}\right) + \cdots + \left(\frac{1}{\binom{10}{3}} - \frac{1}{\binom{11}{3}}\right) \\ &= \frac{1}{\binom{3}{3}} - \frac{1}{\binom{11}{3}} \\ &= \frac{164}{165},\end{aligned}$$

and therefore, our answer is $\boxed{329}$.

7. Let a, b, and c be real numbers such that $a+b+c = \frac{1}{a}+\frac{1}{b}+\frac{1}{c}$ and $abc = 5$. The value of

$$\left(a-\frac{1}{b}\right)^3+\left(b-\frac{1}{c}\right)^3+\left(c-\frac{1}{a}\right)^3$$

can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m+n$.

Answer: 77

Solution: First, we have the identity

$$x^3+y^3+z^3-3xyz = (x+y+z)(x^2+y^2+z^2-xy-yz-zx).$$

Define

$$x = a-\frac{1}{b} \qquad y = b-\frac{1}{c} \qquad z = c-\frac{1}{a}$$

and note that

$$a+b+c = \frac{1}{a}+\frac{1}{b}+\frac{1}{c} \implies a-\frac{1}{b}+b-\frac{1}{c}+c-\frac{1}{a} = x+y+z = 0,$$

so we find

$$x^3+y^3+z^3 = 3xyz.$$

Replacing x, y, and z with their expressions in a, b, and c yields

$$\left(a-\frac{1}{b}\right)^3+\left(b-\frac{1}{c}\right)^3+\left(c-\frac{1}{a}\right)^3 = 3\left(a-\frac{1}{b}\right)\left(b-\frac{1}{c}\right)\left(c-\frac{1}{a}\right).$$

Moreover, notice that

$$\begin{aligned}\left(a-\frac{1}{b}\right)\left(b-\frac{1}{c}\right)\left(c-\frac{1}{a}\right) &= abc-\frac{1}{abc}-a-b-c+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\\ &= abc-\frac{1}{abc}\\ &= 5-\frac{1}{5}\\ &= \frac{24}{5}.\end{aligned}$$

Thus,

$$\left(a-\frac{1}{b}\right)^3+\left(b-\frac{1}{c}\right)^3+\left(c-\frac{1}{a}\right)^3 = 3\cdot\frac{24}{5} = \frac{72}{5}$$

and our answer is $\boxed{77}$.

We note that the ordered triple

$$(a,b,c) = (\sqrt{5}, -\sqrt{5}, -1)$$

satisfies the conditions of the problem, and

$$\left(\sqrt{5}+\frac{1}{\sqrt{5}}\right)^3+(-\sqrt{5}+1)^3+\left(-1-\frac{1}{\sqrt{5}}\right)^3 = \frac{72}{5}$$

in accordance with the above solution.

8. Compute the smallest real value C such that the inequality

$$x^2(1+y)+y^2(1+x) \le \sqrt{(x^4+4)(y^4+4)}+C$$

holds for all real x and y.

Answer: 4

Solution: Consider the vectors $\vec{x} = \langle x^2-2, 2x\rangle$ and $\vec{y} = \langle 2, y^2\rangle$. We have by the triangle inequality that

$$\sqrt{x^4+4}+\sqrt{y^4+4} = ||\vec{x}||+||\vec{y}|| \ge ||\vec{x}+\vec{y}|| = ||\langle x^2, y^2+2x\rangle|| = \sqrt{x^4+y^4+4xy^2+4x^2}.$$

Squaring both sides of the resultant inequality gives

$$x^4+y^4+8+2\sqrt{(x^4+4)(y^4+4)} \ge x^4+y^4+4xy^2+4x^2 \implies xy^2+x^2 \le 2+\frac{1}{2}\sqrt{(x^4+4)(y^4+4)}.$$

Applying the same analysis with the vectors $\langle y^2-2, 2y\rangle$ and $\langle 2, x^2\rangle$ yields

$$yx^2+y^2 \le 2+\frac{1}{2}\sqrt{(x^4+4)(y^4+4)},$$

and adding the last two inequalities gives

$$xy^2+x^2+yx^2+y^2 = x^2(1+y)+y^2(1+x) \le \sqrt{(x^4+4)(y^4+4)}+4.$$

We can observe that setting $x=y=2$ attains equality, so the answer is $C = \boxed{4}$.

9. There is a unique unordered triple (a, b, c) of two-digit positive integers a, b, and c that satisfy the equation

$$a^3+3b^3+9c^3 = 9abc+1.$$

Compute $a+b+c$.

Answer: 113

Solution 1: The restriction of a, b, and c to the two-digit positive integers is what primarily gives us trouble, as we can quickly identify other integrals solutions such as $(a, b, c) = (-2, 0, 1)$ or $(a, b, c) = (4, 3, 2)$ (this one is not as obvious). We'd like to be able to construct larger solutions from smaller ones. The method we choose is to make the function $a^3+3b^3+9c^3-9abc$ a multiplicative function of some objects to which we can associate triples (a, b, c). A well-known multiplicative function is the determinant function on the set of matrices. We associate to the triple (a, b, c) the matrix

$$\begin{bmatrix} a & 3c & 3b \\ b & a & 3c \\ c & b & a \end{bmatrix}$$

because the determinant of this matrix is $a^3+3b^3+9c^3-9abc$. The less obvious idea is that the set of matrices of this form is closed under multiplication and inversion, and these operations preserve the association between ordered triples and matrices (demonstrating this involves a fair amount of computation, which we omit). For example, the matrix associated to $(-2, 0, 1)$ is

$$A = \begin{bmatrix} -2 & 3 & 0 \\ 0 & -2 & 3 \\ 1 & 0 & -2 \end{bmatrix}$$

As it turns out, A^{-1} is the matrix associated to $(4, 3, 2)$. Finally, A^{-2} is the matrix associated to $(52, 36, 25)$, which then must satisfy the given equation. Hence, we have the answer $56+36+25 = \boxed{113}$.

Solution 2: We first recall the (somewhat well-known) factorization

$$x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx).$$

The important step we take is further factoring the right (quadratic) factor above. That is,

$$x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z)$$

where

$$\omega = e^{2i\pi/3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

is a complex (primitive, non-real) cube root of unity. This allows us to write

$$1 = a^3 + 3b^3 + 9c^3 - 9abc = (a + b\sqrt[3]{3} + c\sqrt[3]{9})(a + \omega b\sqrt[3]{3} + \omega^2 c\sqrt[3]{9})(a + \omega^2 b\sqrt[3]{3} + \omega c\sqrt[3]{9}).$$

Now we make some key observations (or rather, observations that will be sufficient to yield an answer):

$$\begin{aligned}
1^2 &= (a + b\sqrt[3]{3} + c\sqrt[3]{9})^2(a + \omega b\sqrt[3]{3} + \omega^2 c\sqrt[3]{9})^2(a + \omega^2 b\sqrt[3]{3} + \omega c\sqrt[3]{9})^2 \\
&= ((a^2 + 6bc) + (3c^2 + 2ab)\sqrt[3]{3} + (b^2 + 2ca)\sqrt[3]{9}) \\
&\cdot ((a^2 + 6bc) + (3c^2 + 2ab)\omega\sqrt[3]{3} + (b^2 + 2ca)\omega^2\sqrt[3]{9}) \\
&\cdot ((a^2 + 6bc) + (3c^2 + 2ab)\omega^2\sqrt[3]{3} + (b^2 + 2ca)\omega\sqrt[3]{9}),
\end{aligned}$$

which comes from expanding each square. If we instead multiply the trinomials in pairs, we have

$$\begin{aligned}
1^2 &= (a + b\sqrt[3]{3} + c\sqrt[3]{9})^2(a + \omega b\sqrt[3]{3} + \omega^2 c\sqrt[3]{9})^2(a + \omega^2 b\sqrt[3]{3} + \omega c\sqrt[3]{9})^2 \\
&= (a^2 + ab\omega\sqrt[3]{3} + ac\omega^2\sqrt[3]{9} + ab\sqrt[3]{3} + b^2\omega\sqrt[3]{9} + 3bc\omega^2 + ac\sqrt[3]{9} + 3bc\omega + 3c^2\omega^2\sqrt[3]{3}) \\
&\cdot (a^2 + ab\omega^2\sqrt[3]{3} + ac\omega\sqrt[3]{9} + ab\sqrt[3]{3} + b^2\omega^2\sqrt[3]{9} + 3bc\omega + ac\sqrt[3]{9} + 3bc\omega^2 + 3c^2\omega\sqrt[3]{3}) \\
&\cdot (a^2 + ab\omega^2\sqrt[3]{3} + ac\omega\sqrt[3]{9} + ab\omega\sqrt[3]{3} + b^2\sqrt[3]{9} + 3bc\omega^2 + ac\omega^2\sqrt[3]{9} + 3bc\omega + 3c^2\sqrt[3]{3}) \\
&= ((a^2 - 6bc) + (3c^2 - 2ab)\omega^2\sqrt[3]{3} + (b^2 - 2ac)\omega\sqrt[3]{9}) \\
&\cdot ((a^2 - 6bc) + (3c^2 - 2ab)\omega\sqrt[3]{3} + (b^2 - 2ac)\omega^2\sqrt[3]{9}) \\
&\cdot ((a^2 - 6bc) + (3c^2 - 2ab)\sqrt[3]{3} + (b^2 - 2ac)\sqrt[3]{9}),
\end{aligned}$$

so now we have found that if (a, b, c) is a solution to the given equation, then two more solutions are

$$(a^2 + 6bc, 3c^2 + 2ab, b^2 + 2ca) \text{ and } (a^2 - 6bc, 3c^2 - 2ab, b^2 - 2ca).$$

Now we are fully equipped to find solutions to the given equation. In particular, we can observe that $(-2, 0, 1)$ is a solution and, plugging these values into the second triple (the one on the right) above, that $(4, 3, 2)$ is a solution as well. Now we can plug these values into the first triple (the one on the left) to find that

$$(4^2 + 6 \cdot 3 \cdot 2, 3 \cdot 2^2 + 2 \cdot 4 \cdot 3, 3^2 + 2 \cdot 2 \cdot 4) = (52, 36, 25)$$

is a solution as well. We are given that this solution is unique, whence this is the only solution for a, b, c two-digit positive integers. The desired answer is then $52 + 36 + 25 = \boxed{113}$.

Note that the key idea of this problem is that $f(x) = a^3 + 3b^3 + 9c^3 - 9abc$ is a multiplicative function of $x = a + b\sqrt[3]{3} + c\sqrt[3]{9}$. In this solution, we have simply shown that squaring x and reciprocating x preserve that $f(x) = 1$. Importantly, don't let the cumbersome algebra above fool you into thinking that this is not a nice problem - this is a very nice problem! Recommended reading: number fields and Dirichlet's unit theorem.

10. For $k \geq 1$, define $a_k = 2^k$. Let

$$S = \sum_{k=1}^{\infty} \cos^{-1}\left(\frac{2a_k^2 - 6a_k + 5}{\sqrt{(a_k^2 - 4a_k + 5)(4a_k^2 - 8a_k + 5)}}\right).$$

Compute $\lfloor 100S \rfloor$.

Answer: 157

Solution: Although neither factor in the denominator factors nicely, we can observe that

$$a_k^2 - 4a_k + 5 = (a_k - 2)^2 + 1,$$

and

$$4a_k^2 - 8a_k + 5 = (2a_k - 2)^2 + 1 = (a_{k+1} - 2)^2 + 1.$$

These equations are nicer than what we started with, but we still have to deal with terms of the form $a_k - 2$. We'd like a nice relation between these numbers, and we have two options:

$$a_{k+1} - 2 = 2(a_k - 2) + 2$$

or

$$a_{k+1} - 2 = (a_k - 2) + a_k.$$

The first relation doesn't really lend itself to anything nice because of the 2 floating on the right side. The second relation, however, leads us to consider

$$a_{k+1} - 2 = \sum_{i=1}^{k} a_i.$$

This is somewhat promising, as the sum we need to evaluate starts at $k = 1$. Before we run with this, however, we should attempt to process the numerator of the fraction into something nice. We could observe the relation between the square of the numerator and the radicand of the denominator by analyzing roots, or we could simply look at the product $(a_k - 2)(2a_k - 2)$:

$$(a_k - 2)(2a_k - 2) = 2a_k^2 - 6a_k + 4$$

So we can write the numerator as $(a_k - 2)(2a_k - 2) + 1$. We are tempted to simply continue our analysis with this in mind. But this nice expression is a red herring (after all, it doesn't really lend itself to any nice geometrical construction, which we'd like to have because of the inverse cosine). The key observation is that

$$mn + 1 = \frac{m^2 + n^2 - (m - n)^2 + 2}{2}.$$

The motivation for this is the introduction of many squares (which are nice in geometry, particularly as it relates to trigonometry). The niceness of this becomes even clearer upon recalling that

$$(2a_k - 2) - (a_k - 2) = a_k.$$

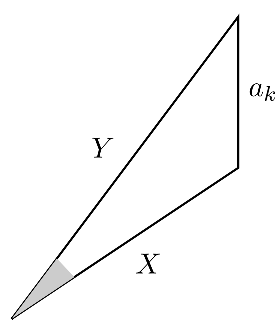
So we (finally) rewrite the sum in question, combining all of our prior observations, as

$$\sum_{k=1}^{\infty} \cos^{-1}\left(\frac{(\sum_{i=1}^{k-1} a_i)^2 + (\sum_{i=1}^{k} a_i)^2 + 2 - a_k^2}{2\sqrt{(1 + (\sum_{i=1}^{k-1} a_i)^2)(1 + (\sum_{i=1}^{k} a_i)^2)}}\right).$$

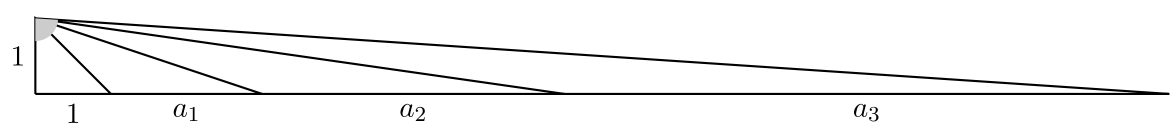
This seems much much worse than what we originally had until we note that, with X as the square root of the left factor in the radical of the denominator and Y as the square root of the right factor, the sum takes the form

$$\sum_{k=1}^{\infty} \cos^{-1}\left(\frac{X^2 + Y^2 - a_k^2}{2XY}\right).$$

The expression inside the parentheses is simply the cosine of the highlighted angle in the following triangle using the Law of Cosines!



and now the problem takes shape. By the Pythagorean theorem, X is the hypotenuse of a right triangle with legs 1 and $\sum_{i=1}^{k-1} a_i$ and Y is the hypotenuse of a right triangle with legs 1 and $\sum_{i=1}^{k} a_i$. This means that the triangles such as the one above fit together as follows (diagram clearly not to scale):



In the triangle above, the angles in our sum are precisely the angles highlighted in gray. The right side of this triangle extends to infinity (as the series with terms a_k diverges), so the shaded angle approaches $\frac{\pi}{2}$. It follows that the original sum equals $\frac{\pi}{2}$ and our answer is $\boxed{157}$.