1. What is the sum of the first 12 positive integers?

   Answer: 78

   Solution: By the formula for the sum of an arithmetic series,

   $$1+2+\cdots+12=\frac{12\cdot 13}{2}=\boxed{78}.$$

2. How many positive integers less than or equal to 100 are multiples of both 2 and 5?

   Answer: 10

   Solution: A positive integer is a multiple of both 2 and 5 if and only if it is a multiple of 10. There are $\frac{100}{10}=\boxed{10}$ positive multiples of 10 less than or equal to 100.

3. Alex has a bag with 4 white marbles and 4 black marbles. She takes 2 marbles from the bag without replacement. What is the probability that both marbles she took are black? Express your answer as a decimal or a fraction in lowest terms.

   Answer: $\frac{3}{14}$

   Solution: The probability of drawing two black marbles without replacement is

   $$\frac{4}{8}\cdot\frac{3}{7}=\boxed{\frac{3}{14}}.$$

4. How many 5-digit numbers are there where each digit is either 1 or 2?

   Answer: 32

   Solution: There are 2 choices for each digit, and there are 5 digits, hence there are $2^5=\boxed{32}$ possible numbers.

5. An integer a with $1\le a\le 10$ is randomly selected. What is the probability that $\frac{100}{a}$ is an integer? Express your answer as decimal or a fraction in lowest terms.

   Answer: $\frac{1}{2}$ OR 0.5

   Solution: Note that $\frac{100}{a}$ is an integer if and only if $a=1,2,4,5,10$. Thus, the probability is $\boxed{1/2}$.

6. Two distinct non-tangent circles are drawn so that they intersect each other. A third circle, distinct from the previous two, is drawn. Let P be the number of points that are on at least 2 circles. How many possible values of P are there?

   Answer: 5

   Solution: The first two circles intersect 2 times by assumption. The third circle can intersect each of the first two 0, 1, or 2 times. Thus we can get 2, 3, 4, 5, or 6 intersection points, giving $\boxed{5}$ possibilities.

7. Let x,y,z be nonzero real numbers such that $x+y+z=xyz$. Compute

   $$\frac{1+yz}{yz}+\frac{1+xz}{xz}+\frac{1+xy}{xy}.$$

   Answer: 4

Solution: Dividing the given equation by xyz gives $\frac{1}{yz} + \frac{1}{xz} + \frac{1}{xy} = 1$. Thus

$$\frac{1+yz}{yz} + \frac{1+xz}{xz} + \frac{1+xy}{xy} = \frac{1}{yz} + \frac{1}{xz} + \frac{1}{xy} + 3 = \boxed{4}.$$

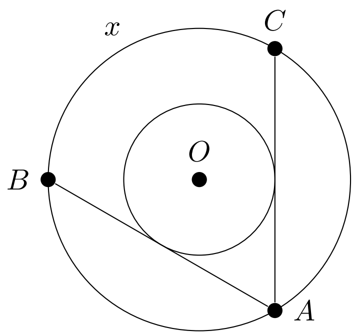
Note that $x = 1$, $y = 2$, and $z = 3$ is one such solution.

8. How many positive integers less than 10^6 are simultaneously perfect squares, cubes, and fourth powers?

   Answer: 3

   Solution: A positive integer is simultaneously a perfect square, cube, and fourth power if and only if it is a perfect 12th power. The number of positive perfect 12th powers less than 10^6 is the same as the number of positive integers less than $\sqrt[12]{10^6} = \sqrt{10}$. Since $3^2 = 9 < 10 < 16 = 4^2$, there are $\boxed{3}$ such integers.

9. Let C_1 and C_2 be two circles centered at point O of radii 1 and 2, respectively. Let A be a point on C_2. We draw the two lines tangent to C_1 that pass through A, and label their other intersections with C_2 as B and C. Let x be the length of minor arc BC, as shown. Compute x.



   Answer: $\mathbf{\frac{4\pi}{3}}$

   Solution: Let X, Y denote the points of tangency of AB, AC with C_1 respectively. Since $OX = 1$ and $OA = 2$, OXA is a 30-60-90 right triangle, $\angle BAC = \angle XAY = 2\angle XAO = 60°$. Thus $\angle BOC = 120°$, so the length of the minor arc BC is

$$\frac{2\pi(2)}{3} = \boxed{\frac{4\pi}{3}}.$$

10. A circle of area π is inscribed in an equilateral triangle. Find the area of the triangle.

    Answer: $\mathbf{3\sqrt{3}}$

    Solution: The inscribed circle has area π and therefore radius 1. The incenter forms a 30-60-90 triangle with a vertex and midpoint of an adjacent side with legs 1 and $\sqrt{3}$. The equilateral triangle is made up of 6 of these right triangles, giving an area of $6 \cdot \frac{\sqrt{3}}{2} = \boxed{3\sqrt{3}}$.

11. Julie runs a 2 mile route every morning. She notices that if she jogs the route 2 miles per hour faster than normal, then she will finish the route 5 minutes faster. How fast (in miles per hour) does she normally jog?

    Answer: 6

Solution: Let t be the amount of time it normally takes Julie to jog her route. Then we have

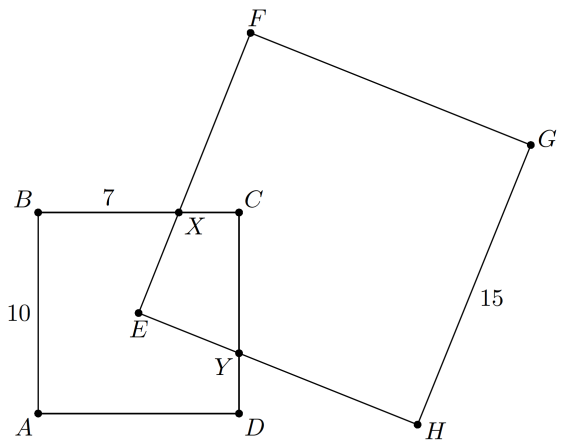
$$\frac{2}{t}+2=\frac{2}{t-\frac{5}{60}}.$$

Multiplying through by $\frac{1}{2}t\left(t-\frac{5}{60}\right)$ gives

$$\begin{aligned} t-\frac{1}{12}+t\left(t-\frac{1}{12}\right)&=t \\ t^2-\frac{1}{12}t-\frac{1}{12}&=0 \\ \left(t-\frac{1}{3}\right)\left(t+\frac{1}{4}\right)&=0. \end{aligned}$$

Since t must be positive, we have $t=\frac{1}{3}$ so that $\frac{2}{t}=\boxed{6}$ miles per hour.

12. Let $ABCD$ be a square of side length 10. Let $EFGH$ be a square of side length 15 such that E is the center of $ABCD$, EF intersects BC at X, and EH intersects CD at Y (shown below). If $BX=7$, what is the area of quadrilateral $EXCY$?



Answer: 25

Solution: Extending lines EX and EY divides the square into four quadrilaterals. Rotating the square $90°$ about E sends the lines EX and EY to each other (since $\angle XEY$ is a right angle), and therefore preserves the four quadrilaterals. It follows that the quadrilaterals are all congruent. Thus the area of each quadrilateral (including $EXCY$) is $\frac{100}{4}=\boxed{25}$. (Note that this solution does not use the fact that $BX=7$.)

13. How many solutions are there to the system of equations

$$\begin{gathered} a^2+b^2=c^2 \\ (a+1)^2+(b+1)^2=(c+1)^2 \end{gathered}$$

if a, b, and c are positive integers?

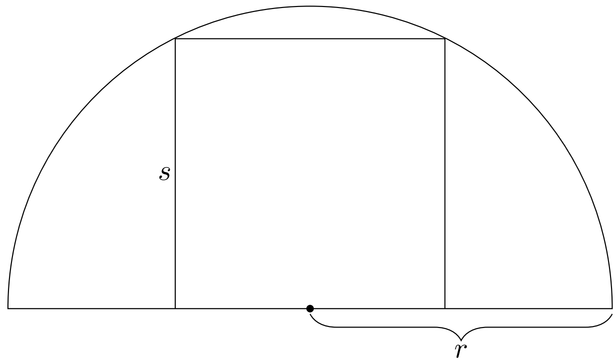
Answer: 0

Solution: Subtracting the first equation from the second gives

$$2a+2b+2-2c=1.$$

Since a, b, and c are integers, the left-hand side is even. But the right-hand side is 1, which is odd, so no such integers a,b,c can exist. Hence there are $\boxed{0}$ solutions.

14. A square of side length s is inscribed in a semicircle of radius r as shown. Compute $\frac{s}{r}$.



Answer: $\frac{2}{\sqrt{5}}$ **OR** $\frac{2\sqrt{5}}{5}$

Solution: Assume without a loss of generality that the radius of the semicircle is 1, so $\frac{s}{r} = s$. Let $ABCD$ denote the square so that segment AB lies on the semicircle. Let O denote the center of the semicircle. By symmetry, the midpoint of AB is the center of the semicircle. If $AB = s$, then $OB = \frac{s}{2}$, $BC = s$, and $OC = 1$ (because OC is a radius of the circle). By the Pythagorean theorem, we have $1 = s^2 + \frac{s^2}{4}$. Hence $s = \boxed{2/\sqrt{5}}$.

15. S is a collection of integers n with $1 \le n \le 50$ so that each integer in S is composite and relatively prime to every other integer in S. What is the largest possible number of integers in S?

Answer: 4

Solution: Taking $S = \{4, 9, 25, 49\}$, we see that $|S| = 4$ is possible. We will show that $|S| = 4$ is also maximal. Indeed, note that if $|S| \ge 5$, then there must be at least 5 primes dividing the elements of S, forcing one such prime factor to be at least 11. As $11 \cdot 5 > 50$, the element in S divisible by 11 must also be divisible by 2 or 3. But then since $|S| \ge 5$ and since one of the elements in S is divisible by two distinct primes, there are 6 distinct prime factors among the elements of S. Hence some element of S is divisible by a prime at least 13, and as before this implies that this element is also divisible by 2 or 3. Again, this implies that there must be 7 primes dividing the elements of S, so that some prime at least 17 divides an element. This element must again be divisible by 2 or 3. This gives a contradiction as we now have 3 elements that are divisible by 2 or 3, so that 2 of these elements must share a factor. Thus, the maximum possible value of $|S|$ is $\boxed{4}$.

16. Let $ABCD$ be a regular tetrahedron and let W, X, Y, Z denote the centers of faces ABC, BCD, CDA, and DAB, respectively. What is the ratio of the volumes of tetrahedrons $WXYZ$ and $WAYZ$? Express your answer as a decimal or a fraction in lowest terms.

Answer: $\frac{1}{2}$ **OR 0.5**

Solution: Note that the two tetrahedra share the base WYZ, so the desired ratio is just the ratio of the heights of the two tetrahedra. Let M denote the midpoint of BC and N the center of triangle WYZ. Since W is the center of ABC, $\frac{AW}{WM} = 2$, so using the similarity of triangles AWN and AMX, we see that the height of $WAYZ$ is twice the height of $WXYZ$. Hence the ratio of the volumes of tetrahedra $WXYZ$ and $WAYZ$ is $\boxed{1/2}$.

17. Consider a random permutation $(s_1, s_2, \ldots, s_8)$ of $(1, 1, 1, 1, -1, -1, -1, -1)$. Let S be the largest of the numbers

$$s_1, \quad s_1 + s_2, \quad s_1 + s_2 + s_3, \quad \ldots, \quad s_1 + s_2 + \cdots + s_8.$$

What is the probability that S is exactly 3? Express your answer as a decimal or a fraction in lowest terms.

Answer: $\frac{1}{10}$ **OR 0.1**

Solution: We claim that if $S = 3$ then exactly one of s_1, s_2, s_3, s_4 is -1. Indeed, if none are -1 then $S \geq s_1 + \cdots + s_4 = 4$, and if two of them are -1 then $S \leq 4 - 2 = 2$. Moreover, by the same reasoning, if $S = 3$ and one of s_1, s_2, s_3 is -1 then $s_5 = 1$. Thus, a permutation with $S = 3$ must start in one of the following ways:

$$\begin{aligned}&1, 1, 1, -1, \ldots\\ &1, 1, -1, 1, 1, \ldots\\ &1, -1, 1, 1, 1, \ldots\\ &-1, 1, 1, 1, 1, \ldots.\end{aligned}$$

It is also clear that any permutation that starts in one of these ways has $S = 3$.

In the first case there are 4 ways to end the permutation since there are four places for the last $+1$. Each of the other three cases can only end in one way since all of the $+1$'s have been used. As there are $\binom{8}{4} = 70$ total permutations, the probability that $S = 3$ is

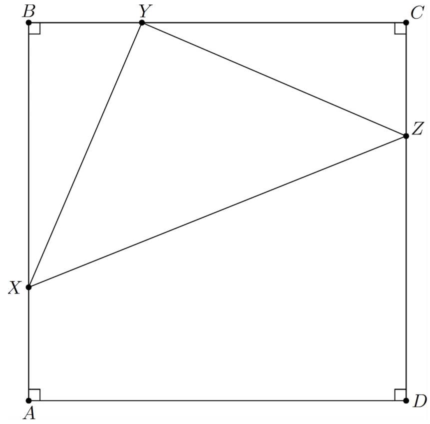
$$\frac{4+1+1+1}{70} = \boxed{\frac{1}{10}}.$$

18. A positive integer is called *almost-kinda-semi-prime* if it has a prime number of positive integer divisors. Given that there are 168 primes less than 1000, how many almost-kinda-semi-prime numbers are there less than 1000?

Answer: 184

Solution: For a number to have p divisors for a prime p, it must be of the form q^{p-1} where q is prime. When $p = 2$, we thus get one almost-kinda-semi-prime for each prime less than 1000, giving 168. For $p = 3$ we get one for every prime q with $q^2 < 1000$. As $31^2 = 961 < 1000 < 32^2$, and there are 11 primes not exceding 31, this gives 11 more almost-kinda-semi-primes. Proceding in this way, we get 3 more when $p = 5$ and 2 more when $p = 7$. As $2^{10} > 1000$, we get no more for $p \geq 11$. So in total we have found $168+11+3+2 = \boxed{184}$ almost-kinda-semi-primes.

19. Let $ABCD$ be a unit square and let X, Y, Z be points on sides AB, BC, CD, respectively, such that $AX = BY = CZ$. If the area of triangle XYZ is $\frac{1}{3}$, what is the maximum value of the ratio XB/AX?



Answer: $\mathbf{\frac{1+\sqrt{3}}{-1+\sqrt{3}}}$ **OR** $\mathbf{2+\sqrt{3}}$

Solution: Let $\min(AX, XB) = r$. Since $AX = CZ$, $BX = DZ$, and $BC = AD$, trapezoids $XADZ$ and $ZCBX$ have equal bases and equal heights, so they have equal area. Now, we can write

$$1 = \text{area}(ABCD) = \text{area}(AXZD)+\text{area}(XYZ)+\text{area}(XBY)+\text{area}(YCZ) = \frac{1}{2}+\frac{1}{3}+r(1-r),$$

and solving for r yields $r = \frac{1}{2} - \frac{\sqrt{3}}{6}$. Hence

$$\frac{AX}{XB} = \frac{1-r}{r} = \frac{\frac{1}{2}+\frac{\sqrt{3}}{6}}{\frac{1}{2}-\frac{\sqrt{3}}{6}} = \boxed{2+\sqrt{3}}.$$

20. Positive integers $a \le b \le c$ have the property that each of $a+b$, $b+c$, and $c+a$ are prime. If $a+b+c$ has exactly 4 positive divisors, find the fourth smallest possible value of the product $c(c+b)(c+b+a)$.

Answer: 50616

Solution: We claim that $(a, b, c) = (1, 1, p-1)$ for some prime p. To see this, note that $a+b$, $b+c$ and $c+a$ are three primes such that their sum

$$(a+b)+(b+c)+(c+a) = 2(a+b+c)$$

is even. Hence either one of these primes is even or all three of them are even. If all three are even then $a = b = c = 1$ (since 2 is the only even prime), giving $a+b+c = 3$, which does not have 4 positive divisors. Hence $a+b = 2$ and the other two primes are odd, giving $a = b = 1$, and $c = p-1$ for some prime p.

Since $a+b+c = p+1$, we need to find primes p such that $p+1$ has exactly 4 divisors. This occurs in two cases: (1) $p+1$ is the cube of a prime, or (2) $p+1$ is a product of two distinct primes.

Case 1: If $p+1 = q^3$ for a prime q, then we have

$$p = q^3 - 1 = (q-1)(q^2+q+1).$$

As p is prime, we have $q = 2$, so that $p = 7$.

Case 2: Suppose $p+1 = qr$ is a product of distinct primes (note that this implies $p \neq 2$). Then $p+1$ is even, and $\frac{p+1}{2}$ is a prime. The first few such p are given by $p = 5, 13, 37, \ldots$.

We want the fourth smallest value of $c(c+b)(c+b+a) = (p-1)p(p+1)$. As the fourth smallest possible value of p is 37, we get an answer of $36 \cdot 37 \cdot 38 = \boxed{50616}$.