GG31 1. True or False? The answer to the question is true.

Answer: True OR False (both are correct. Either one gets full credit)

Solution: The answer being true is consistent with the problem statement. The answer being false is also consistent with the problem statement. Thus $\boxed{\text{true or false}}$ are both correct.

GG33 2. True or False? If p is a prime, m is an integer, and $\frac{m}{p}$ is an integer, then $p = m$.

Answer: False

Solution: Let $p = 2$ and $m = 4$. Then $\frac{m}{p} = 2$ is an integer, but $p \neq m$. So the statement is $\boxed{\text{false}}$.

GG04 3. True or False? If a, b, c, and d are nonzero real numbers where $\frac{a}{b} = \frac{c}{d}$, then $\frac{a}{b} = \frac{a+c}{b+d} = \frac{c}{d}$.

Answer: True

Solution: If $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$. Thus,

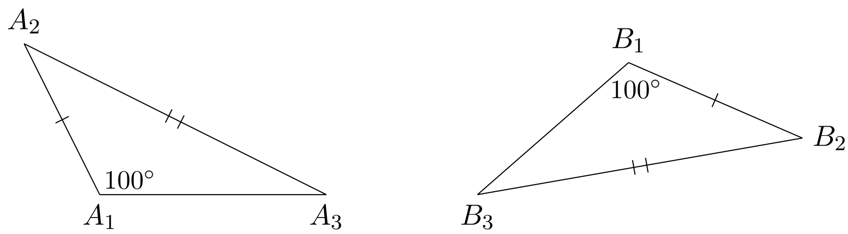
$$a(b + d) = ab + ad = ab + bc = b(a + c),$$

giving $\frac{a}{b} = \frac{a+c}{b+d}$, so the statement is $\boxed{\text{true}}$.

CL06 4. True or False? If $A_1A_2A_3$ and $B_1B_2B_3$ are triangles in which

$$\angle A_2A_1A_3 = \angle B_2B_1B_3 = 100^\circ$$
$$A_1A_2 = B_1B_2$$
$$A_2A_3 = B_2B_3$$

as shown, then triangle $A_1A_2A_3$ is congruent to triangle $B_1B_2B_3$.



Answer: True

Solution: Although side-side-angle congruence does not hold in general, it *does* hold when the two congruent angles are obtuse. As this is the case, the triangles are congruent, and the statement is $\boxed{\text{true}}$.

GG20 5. True or False? If three fair coins are flipped, then the probability of landing two heads and one tails in any order is greater than the probability of landing three heads.

Answer: True

Solution: In the case of two heads and one tails, there are 3 ways to choose which flip is tails. This gives a probability of $\frac{3}{8}$. On the other hand, the probability of three heads is $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$, so the statement is $\boxed{\text{true}}$.

GG19 6. For a real number x, the *floor* of x is denoted $\lfloor x \rfloor$, and defined to be the largest integer less than or equal to x. For example, $\lfloor 3.2 \rfloor = 3$ and $\lfloor -1.5 \rfloor = -2$.

True or False? If x and y are real numbers, then

$$\left\lfloor \frac{\lfloor x \rfloor}{y} \right\rfloor = \left\lfloor \frac{x}{y} \right\rfloor.$$

Answer: False

Solution: Let $x = 1.1$ and $y = 0.01$. Then the left hand side is 100 while the right hand side is 110, so the statement is $\boxed{\text{false}}$.

GG32 7. True or False? If $ABCD$ is a quadrilateral, then the perpendicular bisectors of $ABCD$ all intersect in one point.

Answer: False

Solution: If $ABCD$ is a non-isosceles trapezoid, then the statement is $\boxed{\text{false}}$.

FE02 8. True or False? All positive perfect cubes have an odd number of positive divisors.

Answer: False

Solution: Although $8 = 2^3$ is a positive cubic integer, it has $3+1 = 4$ factors. So the answer is $\boxed{\text{false}}$.

LL32 9. True or False? $2017^{2018} > 2018^{2017}$.

Answer: True

Solution: Since exponential functions grow much faster than polynomials, and since 2017 and 2018 are rather large, we expect the quantity with the larger exponent to be larger. In fact, when $a, b \geq 3$ we have $a^b > b^a$. Applying this here, we see that the statement is $\boxed{\text{true}}$.

GG34 10. True or False? A parallelogram with vertices at $(0, 0)$, (a_1, a_2), (b_1, b_2), and $(a_1 + b_1, a_2 + b_2)$ has area $|a_1b_2 - b_1a_2|$.

Answer: True

Solution: $|a_1b_2 - b_1a_2|$ is the absolute value of the determinant of

$$\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$$

which gives the volume of the parallelogram, so the statement is $\boxed{\text{true}}$.

JN01 11. Compute the product of 9 and 17.

Answer: 153

Solution: We see that $9 \cdot 17 = (13 + 4)(13 - 4) = 13^2 - 4^2 = \boxed{153}$.

DP08 12. What is the smallest positive integer divisible by $1, 2, 3, 4, 5, 6$, and 7?

Answer: 420

Solution: We see that the least common multiple of these numbers is

$$2^2 \cdot 3 \cdot 5 \cdot 7 = \boxed{420}.$$

GG24 13. How many integer solutions are there to the equation $x^3 = 10$?

Answer: 0

Solution: We see that $|x| \leq 2$ since when $|x| \geq 3$ we have $|x^3| \geq 27 > 10$. As none of $x = -2, -1, 0, 1, 2$ are solutions, we get $\boxed{0}$ solutions.

LL28 14. If the complement of an angle is 49.75°, what is the supplement of that angle in degrees? Express your answer as a decimal or a fraction in lowest terms.

Answer: 139.75° OR $\frac{559}{4}^\circ$

Solution: The supplement of an angle is 90° more than the complement of the angle so the supplement is $49.75^\circ + 90^\circ = \boxed{139.75^\circ}$ degrees.

GG14 15. What is the maximum number of 30×30 squares that can be placed in a 100×100 square such that no two of the 30×30 squares overlap and each edge of the 30×30 squares is parallel to an edge of the 100×100 square?

Answer: 9

Solution: We see that we can fit at most three 30×30 squares horizontally side-by-side since four 30×30 squares would take up $4 \cdot 30 = 120$ meters horizontally, which exceeds the available space. The same applies to vertically arranged squares, so that we can fit at most $3 \cdot 3 = 9$ squares. And clearly nine 30×30 squares are possible, as we can arrange them to fill a 90×90 square inside of the 100×100 square. Thus the maximum is $\boxed{9}$.

JL44 16. A medium T-shirt is 10% bigger than a small one, and a large T-shirt is 10% bigger than a medium one. What percent larger is a large T-shirt than a small one?

Answer: 21%

Solution: Let L, M, and S be the sizes of the T-shirts. Then $L = 1.1M$ and $M = 1.1S$, so $L = 1.21S$. In other words, the large T-shirt is $\boxed{21\%}$ larger.

LL21 17. Compute $85 \cdot 93 - 81 \cdot 97$.

Answer: 48

Solution: The expression is the same as $(89^2 - 4^2) - (89^2 - 8^2) = 8^2 - 4^2 = \boxed{48}$.

LL20 18. Compute 115^2.

115^2

Answer: 13225

Solution: For any number of the form $a \cdot 10 + 5$, we compute

$$(a \cdot 10 + 5)^2 = a^2 \cdot 10^2 + a \cdot 2 \cdot 5 \cdot 10 + 5^2 = a(a+1) \cdot 100 + 25.$$

Applying this with $a = 11$, we get $\boxed{13225}$.

LL16 19. How many ways can three runners place in a race if ties are possible?

Answer: 13

Solution: We divide the problem into three cases: no ties, one tie, and all runners tied. In the first case, there are $3! = 6$ ways for the runners to place. If there is one tie, then the tie can be for first or second, and there are three options for the person not in the tie. Thus there 6 ways for this case. Finally, in the case of a three-way tie, there is only one possibility. Thus the total is $6 + 6 + 1 = \boxed{13}$ ways.

LL13 20. At Abe's Pizza, there are 4 choices of toppings, 3 choices of cheese, and 3 choices of crust. Each pizza has exactly 1 topping, 1 cheese, and 1 crust. How many different pizzas can be ordered at Abe's Pizza?

Answer: 36

Solution: By the multiplication principle, the total number of combinations is $4 \cdot 3 \cdot 3 = \boxed{36}$.

LL03 21. Let M be the sum of the first 200 positive odd integers. Let N be the sum of the first 100 positive even integers. Find $M - N$.

Answer: 29900

Solution: Let S be the sum of the first m positive odd integers. We can write

$$S + S = 1 + \cdots + 2m - 1 + 2m - 1 + \cdots + 1 = 2m + 2m + \cdots + 2m$$

where there are m terms in the sum. Thus $S = \frac{1}{2}(2m)m = m^2$. By a similar computation, we see that the sum of the first n even integers is $\frac{1}{2}(2n)(2n+2) = n(n+1)$. Applying this to $m = 200$ and $n = 100$ we get

$$200^2 - 100(101) = 40000 - 10100 = \boxed{29900}.$$

LL30 22. What is the time 500 minutes after 10:30 AM? Make sure to specify whether your answer is AM or PM.

Answer: 6:50 PM OR 18:50

Solution: 500 minutes is equal to 8 hours and 20 minutes. 8 hours after 10:30 AM is 6:30 PM, and 20 minutes after that is $\boxed{\text{6:50 PM}}$.

LL31 23. Compute $6 - 3 + \frac{3}{2} - \frac{3}{4} + \frac{3}{8} - \cdots$.

Answer: 4

Solution: This is a geometric series with first term 6 and common ratio $-\frac{1}{2}$. The sum is thus equal to

$$\frac{6}{1 - (-\frac{1}{2})} = \frac{6}{3/2} = \boxed{4}.$$

LL04 24. If a and b are two positive prime numbers whose squares sum to 173, what is the product of a and b?

Answer: 26

Solution: One of the primes must be 2 because the sum of the squares of two odd numbers is even, but 173 is odd. The other prime is then $\sqrt{173 - 4} = 13$, giving $2 \cdot 13 = \boxed{26}$.

GG27 25. What is the least nonnegative integer n such that $2^{2^n} + 3$ is composite?

Answer: 3

Solution: We compute

$$2^{2^0} + 3 = 5$$
$$2^{2^1} + 3 = 7$$
$$2^{2^2} + 3 = 19$$
$$2^{2^3} + 3 = 259 = 7 \cdot 37$$

giving an answer of $\boxed{3}$.

LL26 26. How many positive factors does 6! have?

Answer: 30

Solution: The prime factorization of 6! is $2^4 \cdot 3^2 \cdot 5$. Thus, it has $(4+1)\cdot(2+1)\cdot(1+1) = \boxed{30}$ factors.

LL01 27. Compute $37^3 + 3 \cdot 37^2 \cdot 63 + 3 \cdot 37 \cdot 63^2 + 63^3$.

Answer: 1000000

Solution: Note that $(a+b)^3 = a^3+3a^2b+3ab^2+b^3$. Thus, $37^3+3\cdot37^2\cdot63+3\cdot37\cdot63^2+63^3 = (37+63)^3 = \boxed{1000000}$.

GG23 28. What is the largest prime number that is a factor of 2018?

Answer: 1009

Solution: We note that $2018 = 2 \cdot 1009$, and that $\boxed{1009}$ is prime.

LL07 29. The greatest common factor of two integers a and b is 22. The least common multiple of a and b is 88. What is the product of a and b?

Answer: 1936

Solution: The product of the LCM and GCF of two numbers is equal to the product of the two numbers. Thus the product of a and b is $22 \cdot 88 = \boxed{1936}$.

LL15 30. If the chairs in an auditorium are organized into equal-length rows of 13 chairs, there are 11 left over. If they are organized into equal-length rows of 7 chairs, there are 5 left over. What is the smallest number of chairs that the auditorium could have?

Answer: 89

Solution: The number of chains is 2 less than a multiple of 13 and also 2 less than a multiple of 7. As the least multiple of 13 and 7 is 91, our answer is $91 - 2 = \boxed{89}$.

LL25 31. What is the maximum area of a rectangle with perimeter 100?

Answer: 625

Solution: The sum of the length and width of the rectangle is $\frac{100}{2} = 50$. For their product to be maximized, the length and width should be equal. Thus they are both 25, giving an area of $25^2 = \boxed{625}$.

LL23 32. What is the area of a circle with circumference 12π?

Answer: 36π

Solution: The circumference is $2\pi r$, so $r = 6$. The area is then $\pi r^2 = \boxed{36\pi}$.

LL27 33. What is the angle, in degrees, formed by the minor arc of the hour and minute hand of a clock at 11:40?

Answer: $110°$

Solution: The minute hand points directly at the 8, so it is $120°$ counterclockwise from pointing straight up. The hour hand is $\frac{2}{3}$ of the way from the 11 to the 12. The angle between two consecutive numbers is $\frac{360}{12} = 30°$, so the hour hand is $10°$ counterclockwise from vertical. The angle formed between them is then $120° - 10° = \boxed{110°}$.

LL02 34. Find the area of an equilateral triangle with side length 10.

Answer: $25\sqrt{3}$

Solution: Consider an equilateral triangle of side length s. An altitude of an equilateral triangle divides the triangle into two congruent $30 - 60 - 90$ triangles. By properties of such triangles, we see that the height of the equilateral triangle is $\frac{s\sqrt{3}}{2}$. Thus the area is $\frac{1}{2} \cdot \frac{s^2\sqrt{3}}{2} = \frac{s^2\sqrt{3}}{4}$. Plugging in $s = 10$ gives $\boxed{25\sqrt{3}}$.

LL17 35. Compute

$$\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{99\cdot 100}.$$

Express your answer as a decimal or a fraction in lowest terms.

Answer: $\mathbf{\frac{49}{100}}$ **or 0.49**

Solution: We can write the fraction $\frac{1}{n\cdot(n+1)}$ as $\frac{1}{n}-\frac{1}{n+1}$. We thus obtain the telescoping sum

$$\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{99}-\frac{1}{100}=\frac{1}{2}-\frac{1}{100}=\boxed{\frac{49}{100}}.$$

LL08 36. For how many integers n is $n^2+11n+30$ a prime number?

Answer: 2

Solution: Note that $n^2+11n+30=(n+5)(n+6)$, which is always even. Thus it is only prime if it is equal to 2, which occurs when n is -4 or -7. Therefore there are $\boxed{2}$ such integers.

DP01 37. Call a number *special* if it has exactly three prime factors. What is the smallest special number over 100? (Note: 1 is not prime.)

Answer: 102

Solution: We note that 101 is prime, and $102=2\cdot 3\cdot 17$. Thus the answer is $\boxed{102}$.

LL10 38. The class average for a math exam was 88 out of 100. After a student with a score of 60 leaves the class, the average rises to 95. How many students are now in the class?

Answer: 4

Solution: Let N be the number of students remaining. The remaining students collectively scored $95N$ points on the exam. Since the previous average was 88, we have

$$\frac{95N+60}{N+1}=88$$

Solving this equation gives $N=\boxed{4}$.

LL11 39. How many diagonals does a regular decagon have? (A diagonal is any segment connecting two vertices that do not share a side, and a decagon has 10 vertices.)

Answer: 35

Solution: In an n-gon, there is a diagonal for every pair of distinct, non-adjacent vertices. Thus we may choose any of the n vertices, and any of the $n-3$ other vertices not adjacent to the first. As the order does not matter, this gives $\frac{n(n-3)}{2}$ such pairs. When $n=10$, we thus get $\frac{10\cdot 7}{2}=\boxed{35}$ diagonals.

LL18 40. What is the smallest positive integer with 8 positive divisors?

Answer: 24

Solution: A number with 8 positive divisors must be of one of the forms pqr, p^3q, or p^7 for distinct primes p, q, and r. Picking the smallest such primes, we have the options $2\cdot 3\cdot 5=30$, $2^3\cdot 3=24$, and $2^7=128$. The smallest of these is $\boxed{24}$.

LL29 41. A math club with 7 girls and 5 boys wants to select a team of 3 girls and 2 boys for BmMT. How many different teams can the math club make?

Answer: 350

Solution: There are $\binom{7}{3}=35$ ways to choose the girls and $\binom{5}{2}=10$ ways to choose the boys. This gives a total of $35\cdot 10=\boxed{350}$ ways to choose a team.

GG26 42. How many pairs of positive integers a and b exist that satisfy $a+b=100$ and $a<b$?

Answer: 49

Solution: Note that if $a+b=100$, then $a<b$ exactly when $a \leq 49$. Moreover, each integer $1 \leq a \leq 49$ has a corresponding $b = 100-a \geq 51 > a$ so that $a+b=100$. Thus there are $\boxed{49}$ solutions.

LL22 43. 10 distinct circles are drawn in the plane. What is the maximum number of points on an intersection between two or more circles?

Answer: 90

Solution: There are $\binom{10}{2} = 45$ pairs of circles, and each pair of circles can intersect at most 2 times. Thus the maximumum number of intersections is $45 \cdot 2 = \boxed{90}$.

One way you can do this is as follows: Draw one circle of radius 1. Select 9 points equally spaced around the circumference of the circle, and draw a new circle centered at each point of radius 1.5.

GG25 44. How many prime numbers are there between 50 and 100?

Answer: 10

Solution: The prime numbers between 50 and 100 are

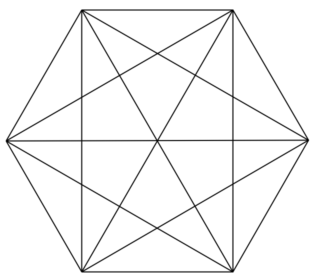
$$53, 59, 61, 67, 71, 73, 79, 83, 89, \text{ and } 97.$$

This gives $\boxed{10}$ such numbers.

LL24 45. All diagonals in a regular hexagon are drawn. How many points inside the hexagon are the intersection of two or more diagonals? (A diagonal is any segment connecting two vertices that do not share a side.)

Answer: 13

Solution: Drawing the diagonals of a hexagon, as shown, and counting intersections gives $\boxed{13}$ points.



LL19 46. Express $0.5\overline{72}$ as a fraction in lowest terms.

Answer: $\mathbf{\frac{63}{110}}$

Solution: Since $0.\overline{72} = \frac{72}{99}$, we have that

$$0.5\overline{72} = 0.5 + 0.0\overline{72} = \frac{1}{2} + \frac{1}{10} \cdot \frac{72}{99} = \boxed{\frac{63}{110}}.$$

LL09 47. How many ways are there to make 88 cents using only pennies, nickels, and dimes? (Pennies are worth 1 cent, nickels are worth 5 cents, and dimes are worth 10 cents.)

Answer: 90

Solution: We first set the number of nickels and dimes, and then add whatever number of pennies is necessary to make 88 cents. If we have d dimes with $0 \le d \le 8$, then we can have n nickels for any $0 \le n \le \frac{85-10d}{5} = 17 - 2d$; this gives $18 - 2d$ ways to make 88 cents when using d dimes. This the total number of ways to make 88 cents is

$$\sum_{d=0}^{8} 18 - 2d = 18 + 16 + \cdots + 2 = \boxed{90}.$$

LL05 48. A 6-sided die is rolled 3 times. What is the probability that the product of the rolls is prime? Express your answer as a decimal or a fraction in lowest terms.

Answer: $\frac{1}{24}$ or $0.041\overline{6}$

Solution: Two of the rolls must be 1, and the other roll must be prime. The roll that is not 1 can be 2, 3, or 5. There are 3 ways to choose the roll that is not 1. Thus our answer is

$$\frac{3 \cdot 3}{6 \cdot 6 \cdot 6} = \boxed{\frac{1}{24}}.$$

JL45 49. BmMT needs to buy breakfast for its staff. The local bagel shop sells bags of 12 bagels. If each bagel can be either a plain bagel, sesame bagel, or garlic bagel, how many different bags of bagels can BmMT buy?

Answer: 91

Solution: Each bag consists of 12 bagels, each of which may have 3 possible flavors, and order does not matter. By the stars-and-bars method of counting, this gives $\binom{12+3-1}{3-1} = \boxed{91}$ possible bags.

LL14 50. How many positive divisors of 10! are perfect cubes?

Answer: 6

Solution: We first factorize 10! to get $10! = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7^1$. For a number to be a perfect cube, the exponent of each prime in its factorization must be a multiple of 3. Thus the perfect cube factors of 10! are of the form

$$2^a \cdot 3^b \cdot 5^c \cdot 7^d \quad \text{with } a = 0, 3, 6, \;\; b = 0, 3, \;\; \text{and } \; c = d = 0.$$

This gives $3 \cdot 2 = \boxed{6}$ such factors.