1. A bus leaves San Mateo with n fairies on board. When it stops in San Francisco, each fairy gets off, but for each fairy that gets off, n fairies get on. Next it stops in Oakland where 6 times as many fairies get off as there were in San Mateo. Finally the bus arrives at Berkeley, where the remaining 391 fairies get off. How many fairies were on the bus in San Mateo?

   Answer: 23

   Solution: Initially, there were n fairies. After the first bus stop in San Francisco, there were $n \cdot n = n^2$ faries. At Oakland, $6n$ fairies get off, so there are $n^2 - 6n$ fairies left. At Berkeley, the remaining 391 get off, so $n^2 - 6n = 391$. Taking the positive solution to the quadratic equation $n^2 - 6n - 391$ gives $n = \boxed{23}$.

2. Let a and b be two real solutions to the equation $x^2 + 8x - 209 = 0$. Find $\frac{ab}{a+b}$. Express your answer as a decimal or a fraction in lowest terms.

   Answer: $\mathbf{\frac{209}{8}}$

   Solution: A quadratic polynomial with roots a and b and leading coefficient 1 is given by

$$(x-a)(x-b) = x^2 - (a+b)x + ab.$$

   We find from here that $a + b = -8$ and $ab = -209$. Hence

$$\frac{ab}{a+b} = \frac{-209}{-8} = \boxed{\frac{209}{8}}.$$

3. Let a, b, and c be positive integers such that the least common multiple of a and b is 25 and the least common multiple of b and c is 27. Find abc.

   Answer: 675

   Solution: Since $\text{lcm}(a, b) = 25$ and $\text{lcm}(b, c) = 27$, b must divide both 25 and 27. But 25 and 27 have no common factors, so b must equal 1. Therefore $a = 25$ and $c = 27$. This gives that $abc = 25 \cdot 27 = \boxed{675}$

4. It takes Justin 15 minutes to finish the Speed Test alone, and it takes James 30 minutes to finish the Speed Test alone. If Justin works alone on the Speed Test for 3 minutes, then how many minutes will it take Justin and James to finish the rest of the test working together? Assume each problem on the Speed Test takes the same amount of time.

   Answer: 8

   Solution: In 3 minutes, Justin completes $\frac{1}{5}$ of the test, with $\frac{4}{5}$ of the test remaining. Justin and James together can solve $\frac{1}{15} + \frac{1}{30} = \frac{1}{10}$ of the test per minute working together. Thus the rest of the test takes them

$$\frac{4/5}{1/10} = \boxed{8}$$

   minutes.

5. Angela has 128 coins. 127 of them have the same weight, but the one remaining coin is heavier than the others. Angela has a balance that she can use to compare the weight of two collections of coins against each other (that is, the balance will not tell Angela the weight of a collection of coins, but it will say which of two collections is heavier). What is the minumum number of weighings Angela must perform to guarantee she can determine which coin is heavier?

   Answer: 5

   Solution: We can do this in 5 weighings in the following manner: Split the 128 coins into three groups of size 42/43/43. Weigh the two groups of size 43 against each other. If one side is heavier, then the heavy coin is on the heavier side; if the scale weighs equal, then the heavy coin is in the group of size 42. For weighing number 2, split the group of 42 into 14/14/14, or the group of 43 into 14/14/15 (whichever contains the heavy coin), and weight them in the same manner as the previous weighing. For weighing number 3, split the group of 14 into 4/5/5, or the group of 15 into 5/5/5. For weighing number 4, split the group of 4 into 1/1/2, or the group of 5 into 1/2/2. If a group of 1 contains the heavy coin, then we have found the heavy coin. If a group of 2 contains the heavy coin, then weigh the two coins and pick the heavy coin.

We will now show that 4 weighings is insufficient. Note that each weighing can produce 3 possible outcomes: the left is heavier, the right is heavier, or the sides are equal weight. Thus with 4 weighings, there are $3^4 = 81$ possible outcomes. As there are $128 > 81$ coins that could be the heavy coin, 4 weighings therefore cannot distinguish all of the possibilities, as claimed.