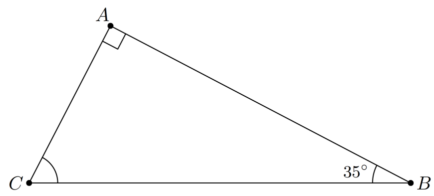
1. If x is a real number that satisfies $\frac{48}{x} = 16$, find the value of x.

   Answer: 3

   Solution: We get $x = \frac{48}{16} = \boxed{3}$.

2. If ABC is a right triangle with hypotenuse BC such that $\angle ABC = 35°$, what is $\angle BCA$ in degrees?



   Answer: 55°

   Solution: We see that $\angle BCA = 180° - 90° - 35° = 180° - 125° = \boxed{55°}$.

3. If $a \triangle b = a + b - ab$, find $4 \triangle 9$.

   Answer: −23

   Solution: We see that $4 \triangle 9 = 4 + 9 - 4 \cdot 9 = 13 - 36 = \boxed{-23}$.

4. Grizzly is 6 feet tall. He measures his shadow to be 4 feet long. At the same time, his friend Panda helps him measure the shadow of a nearby lamp post, and it is 6 feet long. How tall is the lamp post in feet?

   Answer: 9

   Solution: $\frac{6}{4} = \frac{x}{6}$, where x is the height of the lamp post, so $x = \boxed{9}$.

5. Jerry is currently twice as old as Tom was 7 years ago. Tom is 6 years younger than Jerry. How many years old is Tom?

   Answer: 20

   Solution: Let T denote Tom's age, and let J denote Jerry's age. We have

$$J = 2(T - 7)$$
$$T = J - 6.$$

   Substituting the second equation into the first gives $J = 2(J - 13)$, so that $J = 26$. Hence $T = \boxed{20}$.

6. Out of the 10,000 possible four-digit passcodes on a phone, how many of them contain only prime digits?

   Answer: 256

   Solution: There are four one-digit prime numbers: 2, 3, 5, and 7. Thus there are $4^4 = \boxed{256}$ such passcodes.

7. It started snowing, which means Moor needs to buy snow shoes for his 6 cows and 7 sky bison. A cow has 4 legs, and a sky bison has 6 legs. If Moor has 36 snow shoes already, how many more shoes does he need to buy? Assume cows and sky bison wear the same type of shoe and each leg gets one shoe.

   Answer: 30

   Solution: Since a cow has 4 legs, and Moor has 6 cows, Moor needs $6 \cdot 4 = 24$ snow shoes for his cows. Similarly, Moor needs $6 \cdot 7 = 42$ snow shoes for his sky bison. Since Moor has 36 snow shoes already, he only needs to buy $42 + 24 - 36 = \boxed{30}$ more snow shoes.

8. How many integers n with $1 \leq n \leq 100$ have exactly 3 positive divisors?

   Answer: 4

   Solution: An integer has exactly 3 positive divisors if and only if it is the square of a prime. There are exactly $\boxed{4}$ such numbers less than 100: 4, 9, 25, and 49.

9. James has 3 red candies and 3 green candies. 3 people come in and each randomly take 2 candies. What is the probability that no one got 2 candies of the same color? Express your answer as a decimal or a fraction in lowest terms.

   Answer: $\frac{2}{5}$ or 0.4

   Solution: The first person picks a candy; without loss of generality let it be red. They then have a $\frac{3}{5}$ chance of picking a green candy for their second candy. The second person picks a candy (without loss of generality let it be red as well); they then have a $\frac{2}{3}$ chance of picking a green candy. Finally, the last person picks a candy, and they are assured (if the other two steps have gone right) to pick the opposite color for the right candy.

   The probability of this happening is

$$\frac{3}{5} \cdot \frac{2}{3} = \boxed{\frac{2}{5}}.$$

10. When Box flips a *strange coin*, the coin can land heads, tails, or on the side. It has a $\frac{1}{10}$ probability of landing on the side, and the probability of landing heads equals the probability of landing tails. If Box flips a strange coin 3 times, what is the probability that the number of heads flipped is equal to the number of tails flipped? Express your answer as a decimal or a fraction in lowest terms.

    Answer: $\frac{49}{400}$ or 0.1225

    Solution: The probability that the coin lands heads is the same as the probability that it lands tails, which is $\frac{1-\frac{1}{10}}{2} = \frac{9}{20}$. In order for a strange coin to land heads and tails an equal number of times, it can either land on heads once, tails once, and on its side once, or it can land on the side 3 times. In the first case, considering the fact that there are $3! = 6$ ways to order the flips, we get a probability of

$$6 \cdot \frac{1}{10} \cdot \frac{9}{20} \cdot \frac{9}{20} = \frac{243}{2000}.$$

    For the second case, the probability is $\left(\frac{1}{10}\right)^3 = \frac{1}{1000}$. Overall, we get a probability of

$$\frac{243}{2000} + \frac{1}{1000} = \boxed{\frac{49}{400}}.$$

11. James is travelling on a river. His canoe goes 4 miles per hour upstream and 6 miles per hour downstream. He travels 8 miles upstream and then 8 miles downstream (to where he started). What is his average speed, in miles per hour? Express your answer as a decimal or a fraction in lowest terms.

    Answer: 4.8 or 24/5

    Solution: It takes $\frac{8}{4} = 2$ hours for the canoe to travel upstream, and $\frac{8}{6} = \frac{4}{3}$ hours for the canoe to travel downstream. The total distance is $8+8 = 16$ miles. Hence, the average speed is

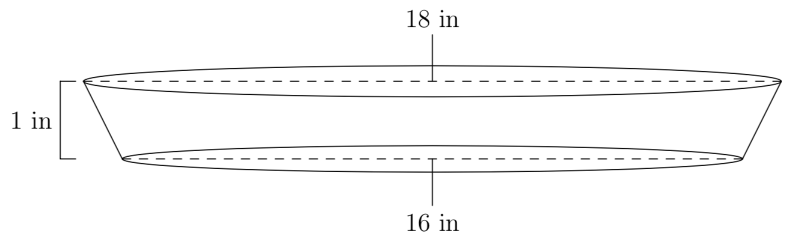
$$\frac{16}{2+\frac{4}{3}} = \boxed{\frac{24}{5}}.$$

12. Four boxes of cookies and one bag of chips cost exactly 1000 jelly beans. Five bags of chips and one box of cookies cost less than 1000 jelly beans. If both chips and cookies cost a whole number of jelly beans, what is the maximum possible cost of a bag of chips?

Answer: 156

Solution: Let x be price of a box of cookies, and let y be the price of a bag of chips. Then $4x + y = 1000$, and $x + 5y < 1000$. Solving for y, we get $y < \frac{3000}{19}$. Since y must be a whole number, $y \le 157$. We must also have that $x = \frac{1000-y}{4}$ is an integer, which occurs if and only if y is a multiple of 4. Thus the maximum value of y is the largest multiple of 4 less than or equal to 157. Hence, the answer is $\boxed{156}$.

13. June is making a pumpkin pie, which takes the shape of a truncated cone, as shown below. The pie tin is 18 inches wide at the top, 16 inches wide at the bottom, and 1 inch high. How many cubic inches of pumpkin filling are needed to fill the pie?



Answer: $\frac{217\pi}{3}$

Solution: We claim that the pumpkin pie is a cone with a base of radius 9 and a height of 9, minus a cone with a base of radius 8 and a height of 8. To see this, we focus on the cone from which the pumpkin pie was truncated. Let A be the center of the top face of the pie, let B be the center of the bottom face of the pie, and let X be the tip of the cone. Since the pumpkin pie is 1 inch high, we have $AX = BX + 1$. By similar triangles, we have

$$\frac{9}{8} = \frac{AX}{BX} = \frac{BX+1}{BX}.$$

Solving for BX, we get $BX = 8$ and $AX = 9$ as claimed. Thus, the volume of the pumpkin pie is

$$\frac{1}{3}\pi \cdot 9^2 \cdot 9 - \frac{1}{3}\pi \cdot 8^2 \cdot 8 = \boxed{\frac{217\pi}{3}}$$

cubic inches.

14. For two real numbers a and b, let $a\#b = ab - 2a - 2b + 6$. Find a positive real number x such that $(x\#7)\#x = 82$.

Answer: 6

Solution: We compute $x\#7 = 7x - 2x - 14 + 6 = 5x - 8$. Thus,

$$(5x-8)\#x = 5x^2 - 8x - 10x + 16 - 2x + 6 = 5x^2 - 20x + 22.$$

Therefore $5x^2 - 20x + 22 = 82$ so $x^2 - 4x - 12 = 0$ which has positive solution $x = \boxed{6}$.

15. Find the sum of all positive integers n such that

$$\frac{n^2 + 20n + 51}{n^2 + 4n + 3}$$

is an integer.

Answer: 26

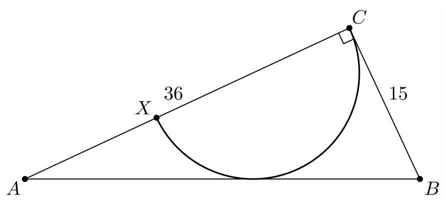
Solution: We see that

$$\frac{n^2+20n+51}{n^2+4n+3} = \frac{(n+3)(n+17)}{(n+3)(n+1)} = \frac{n+17}{n+1}$$

since $n+3 \neq 0$ when n is positive. Thus we want to find all integers n so that $n+1$ divides $n+17$. Setting $m = n+1$, this is equivalent to finding the values $m \geq 2$ so that m divides $m+16$. This occurs if and only if m divides 16, giving $m = 2, 4, 8, 16$. These correspond to $n = 1, 3, 7, 15$, giving a sum of

$$1+3+7+15 = \boxed{26}.$$

16. Let ABC be a right triangle with hypotenuse AB such that $AC = 36$ and $BC = 15$. A semicircle is inscribed in ABC as shown, such that the diameter XC of the semicircle lies on side AC and that the semicircle is tangent to AB. What is the radius of the semicircle?



Answer: 10

Solution: By the Pythagorean theorem, $AB = \sqrt{36^2+15^2} = 39$. Let O be the center of the semicircle, and let D be the point of tangency of the semicircle and AB. By SSA congruence of right triangles, right triangles ODB and OCB are congruent, and therefore, $DB = CB = 15$, and $AD = AB - DB = 24$. We can also see that triangles AOD and ABC are similar. Thus $\frac{24}{36} = \frac{r}{15}$, giving $r = \boxed{10}$.

17. Let a and b be relatively prime positive integers such that the product ab is equal to the least common multiple of 16500 and 990. If $\frac{16500}{a}$ and $\frac{990}{b}$ are both integers, what is the minimum value of $a+b$?

Answer: 599

Solution: Note that $16500 = 2^2 \cdot 3 \cdot 5^3 \cdot 11$, and $990 = 2 \cdot 3^2 \cdot 5 \cdot 11$, so the least common multiple is $L = 2^2 \cdot 3^2 \cdot 5^3 \cdot 11$. Since a and b must be relatively prime, one of a or b must be divisible by 5^3. If b was divisible by 5^3, then $990/b$ would not be an integer, so a must be divisible by 5^3. Similarly, we can show that a is divisible by 2^2 and b is divisible by 3^2. Therefore, the only two possibilities are:

$$a = 2^2 \cdot 5^3 \cdot 11, \quad b = 3^2 \quad \text{or} \quad a = 2^2 \cdot 5^3, \quad b = 3^2 \cdot 11.$$

We see that $a+b$ is minimized in the second case, giving $a+b = \boxed{599}$.

18. Let x be a positive real number so that $x - \frac{1}{x} = 1$. Compute $x^8 - \frac{1}{x^8}$.

Answer: $21\sqrt{5}$

Solution 1: We see that $x^2 - x - 1 = 0$, so that taking the positive solution to the quadratic equation gives

$$x = \frac{1+\sqrt{5}}{2}.$$

Let $F_n = x^n - \frac{1}{x^n}$. We see that $(x + x^{-1})F_n = F_{n+1} + F_{n-1}$. Since

$$x + \frac{1}{x} = \frac{1+\sqrt{5}}{2} + \frac{-1+\sqrt{5}}{2} = \sqrt{5}$$

we therefore find for $n \geq 2$

$$F_{n+1} = \sqrt{5}F_n - F_{n-1}.$$

We have $F_1 = 1$ and $F_2 = \frac{3+\sqrt{5}}{2} - \frac{3-\sqrt{5}}{2} = \sqrt{5}$, so we can recursively compute

$$\begin{aligned}
F_3 &= 4 \\
F_4 &= 3\sqrt{5} \\
F_5 &= 11 \\
F_6 &= 8\sqrt{5} \\
F_7 &= 29 \\
F_8 &= \boxed{21\sqrt{5}}.
\end{aligned}$$

Solution 2: As above we find $x = \frac{1+\sqrt{5}}{2}$ so that $x + x^{-1} = \sqrt{5}$. Thus we can compute

$$\begin{aligned}
x^2 + x^{-2} &= \left(x + x^{-1}\right)^2 - 2 = 3 \\
x^4 + x^{-4} &= \left(x^2 + x^{-2}\right)^2 - 2 = 7.
\end{aligned}$$

Hence

$$x^8 - x^{-8} = \left(x^4 + x^{-4}\right)\left(x^2 + x^{-2}\right)\left(x + x^{-1}\right)\left(x - x^{-1}\right) = 7 \cdot 3 \cdot 1 \cdot \sqrt{5} = \boxed{21\sqrt{5}}.$$

19. Six people sit around a round table. Each person rolls a standard 6-sided die. If no two people sitting next to each other rolled the same number, we will say that the roll is *valid*. How many different rolls are valid?

    Answer: 15630

    Solution: Label each person a number between 1 and 6. It is useful to consider persons 1, 3, and 5.

    Case 1: $1, 3, 5$ **rolled the same number**

    There are 6 numbers they could have rolled, and $5^3 = 125$ possibilities for the other 3 people. This gives $6 \cdot 5^3 = 750$ rolls in this case.

    Case 2: $1, 3, 5$ **rolled** 2 **different numbers**

    There are 3 ways to pick who rolled the number different than the other two, and then $6 \cdot 5$ ways to pick the two numbers. Then there are $5 \cdot 4 \cdot 4$ ways to pick what players 2, 4, and 6 rolled. This gives $3 \cdot 6 \cdot 5^2 \cdot 4^2 = 7200$ rolls.

    Case 3: $1, 3, 5$ **rolled distinct numbers**

    There are $6 \cdot 5 \cdot 4$ ways to pick what players 1, 3, and 5 rolled. Then there are 4^3 ways to pick the other rolls. This gives $6 \cdot 5 \cdot 4^4 = 7680$ rolls.

    All together, we have $750 + 7200 + 7680 = \boxed{15630}$ valid rolls.

20. Given that $\frac{1}{31} = 0.\overline{a_1a_2a_3a_4a_5\cdots a_n}$ (that is, $\frac{1}{31}$ can be written as the repeating decimal expansion $0.a_1a_2\cdots a_na_1a_2\cdots a_na_1a_2\cdots$), what is the minimum value of n?

    Answer: 15

Solution 1: If n is such that $\frac{1}{31}$ has a repeating decimal expansion of this form, then we have

$$\frac{1}{31} = \frac{a}{10^n - 1}$$

where a is the number with digits $a_1 a_2 \cdots a_n$. This occurs if and only if $10^n - 1 = 31a$ for some a, which occurs if and only if $10^n \equiv 1 \pmod{31}$. Thus we want to find the smallest n so that $10^n \equiv 1 \pmod{31}$.

By Fermat's Little Theorem, $10^{30} \equiv 1 \pmod{31}$, so that we must have n dividing 30. But we can compute

$$\begin{aligned}
10^1 &\equiv 10 && \pmod{31} \\
10^2 &\equiv 7 && \pmod{31} \\
10^3 &\equiv 10 \cdot 10^2 \equiv 70 \equiv 8 && \pmod{31} \\
10^5 &\equiv 10^2 \cdot 10^3 \equiv 56 \equiv 25 && \pmod{31} \\
10^6 &\equiv (10^3)^2 \equiv 64 \equiv 2 && \pmod{31} \\
10^{10} &\equiv (10^5)^2 \equiv 25^2 \equiv 625 \equiv 5 && \pmod{31} \\
10^{15} &\equiv 10^5 \cdot 10^{10} \equiv 5 \cdot 25 \equiv 125 \equiv 1 && \pmod{31}
\end{aligned}$$

Hence, the minimal k is $\boxed{15}$.

Solution 2: Note that n is the minimal positive integer such that if we shift the decimal expansion of $\frac{1}{31}$ by n places, then the decimal part of the resulting real number would be the same as the decimal part of $\frac{1}{31}$. But shifting by n places is the same as multiplying by 10^n. Thus, when we multiply $\frac{1}{31}$ by 10^n, the mixed fraction we get is $a\frac{1}{31}$ for some a. Hence

$$\begin{aligned}
10^n \cdot \frac{1}{31} &= a + \frac{1}{31} \\
10^n &= 31a + 1 \\
10^n &\equiv 1 \pmod{31}
\end{aligned}$$

So we must find the minimal n such that $10^n \equiv 1 \pmod{31}$. We may now proceed as in solution 1.