1. True or False? Your answer to this question will be True.

   Answer: True or False (both are correct!)

   Solution: If your answer is True, then the statement was correct in saying your answer would be True. If you put False, then the statement was false, so your answer is correct regardless.

2. True or False? If two polygons are equiangular, then they are similar.

   Answer: False

   Solution: No, both a square and a rectangle have only right angles, but they are not similar because the ratios of their side lengths are different.

3. True or False? $5^9 > 9!$.

   Answer: True

   Solution: We note that we can break $9!$ into the product of $1 \cdot 9$, $2 \cdot 8$, $3 \cdot 7$, $4 \cdot 6$, and 5. We note that $1 \cdot 9$, $2 \cdot 8$, $3 \cdot 7$, and $4 \cdot 6$ are each less than 5^2, so 5^9 must be greater than $9!$.

4. True or False? If a positive integer x has exactly 3 distinct divisors, it must be a perfect square.

   Answer: True

   Solution: Every prime number has exactly 2 divisors, and 1 has exactly one divisor. If x had at least 2 distinct prime factors a and b, then 1, a, b, and ab would all be distinct divisors, so x cannot have more than one prime factor. If $x = a^n$ for some prime a and $n > 2$, then $1, a, a^2, a^3$ would all be divisors. Thus, the only condition for x to have exactly 3 divisors is if $x = a^2$ for some prime a, and its factors would be $1, a, a^2$.

5. True or False? If x is a perfect square, it must have exactly 3 distinct divisors.

   Answer: False

   Solution: If x is not the square of a prime, it will have more than 3 divisors. For example, $4^2 = 16$ has divisors 1, 2, 4, 8, and 16.

6. True or False? Let $f(x, y) = 4x^2 + 12xy + 9y^2 + 1$. There exist real numbers a and b such that $f(a, b) = 0$.

   Answer: False

   Solution: We note that $f(x, y)-1 = (2x+3y)^2$. Since the right hand side is always nonnegative (being a perfect square), we must have $f(x, y) \geq 1$, so there are no a and b such that $f(a, b) = 0$.

7. True or False? If you buy two 12-inch diameter pizzas, you get more pizza than if you buy one 17-inch diameter pizza.

   Answer: False

   Solution: The area of a circle with diameter 12 (radius 6) is $\pi r^2 = 36\pi$, so two of these would have area 72π. This is less than the area of a circle with diameter 17 (radius 8.5), which has area $8.5^2\pi = 72.25\pi$.

8. True or False? When flipping an unbiased coin 10 times, the sequence $HTHHTHTHTH$ is more likely to occur than the sequence $HHHHHHHHHH$.

   Answer: False

   Solution: Both are equally likely.

9. True or False? Out of the integers in the range from 1 to 16 inclusive, more numbers are prime than are perfect squares or perfect cubes.

Answer: True

Solution: There are 6 primes in this range: $2, 3, 5, 7, 11, 13$. There are 4 perfect squares (1^2 to 4^2) and 2 perfect cubes (1^3 and 2^3). 1 is counted twice, so there are 5 numbers that are either perfect squares or perfect cubes, and thus there are more primes.

10. True or False? The probability of rolling 2 dice and getting a sum of at least 7 is greater than the probability of flipping 6 coins and getting at least 3 heads.

Answer: False

Solution: Let p_1 be the probability of rolling a 7. We note that the probability of rolling higher than a 7 is the same as rolling lower than a 7, so the probability of rolling at least a 7 is simply $p_1 + \frac{1-p_1}{2} = \frac{p_1+1}{2}$.

Let p_2 be the probability of getting exactly 3 heads. The probability of getting more than 3 heads is exactly the same as getting less than 3 heads, so by the same reasoning, the probability of getting at least 3 heads is $\frac{p_2+1}{2}$.

We then simply need to compute p_1 and p_2 to determine which one is greater. $p_1 = \frac{1}{6}$, while $p_2 = \binom{6}{3}2^{-6} = \frac{20}{64}$. We can see that $p_2 > p_1$, so the statement is False.

11. Compute $1 + 1$.

Answer: 2

Solution: Observe that $1 + 1 = \boxed{2}$.

12. Compute 29×31.

Answer: 899

Solution: We compute that $29 \times 31 = (30 - 1)(30 + 1) = 30^2 - 1^2 = \boxed{899}$.

13. Compute $1 + 3 - 5 + 7 - 9 + 11 - 13 + 15 - 17$.

Answer: −7

Solution: We see that we can break the terms after the 1 into pairs, each of which are -2, so our answer is $1 - 2 \cdot 4 = \boxed{-7}$.

14. Compute $1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19$.

Answer: 100

Solution: The sum of this series of the first n odd numbers is n^2. Therefore the sum here is $10^2 = \boxed{100}$.

15. Compute $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{32}$.

Answer: 2

Solution: If we first add the two $\frac{1}{32}$ terms we get $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{16}$. If we combine the two $\frac{1}{16}$ terms, we get $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8}$. Continuing in this way, we see that the answer is $1 + 1 = \boxed{2}$.

16. Compute $\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \frac{1}{42}$.

Answer: $\frac{6}{7}$

Solution: Note that the denominators of our fractions are all of the form $n(n+1)$, so we see that $\frac{1}{2}+\frac{1}{6}=\frac{2}{3}$, $\frac{2}{3}+\frac{1}{12}=\frac{3}{4}$, and $\frac{3}{4}+\frac{1}{20}=\frac{4}{5}$. Continuing this pattern gives us our final answer of $\boxed{\dfrac{6}{7}}$.

17. Compute $\sqrt{15}\sqrt{21}\sqrt{35}$.

    Answer: 105

    Solution: Note that $\sqrt{15}\sqrt{21}\sqrt{35}=\sqrt{3^2 5^2 7^2}=3\times 5\times 7=\boxed{105}$.

18. Compute $1^2+2^2+3^2+\cdots+12^2$.

    Answer: 650

    Solution: Recall that

$$1^2+2^2+\cdots+n^2=\frac{n\cdot(n+1)\cdot(2n+1)}{6}.$$

Therefore, we have that

$$1^2+2^2+\cdots+12^2=\frac{12\cdot 13\cdot 25}{6}=\boxed{650}.$$

19. Compute $(-1)^0+(-1)^1+(-1)^2+\cdots+(-1)^{2016}$.

    Answer: 1

    Solution: Note that $(-1)^n$ is $+1$ if n is even and -1 if n is odd. Therefore, this sum is $1-1+1-1+1-\cdots+1=\boxed{1}$.

20. Compute $20^3+19^3+\cdots+16^3$.

    Answer: 29700

    Solution: Recall that $1^3+2^3+\cdots+n^3=(1+2+\cdots+n)^2$. We can use this to compute

$$(1^3+2^3+\cdots+20^3)-(1^3+2^3+\cdots+15^3)=(1+2+\cdots+20)^2-(1+2+\cdots+15)^2$$
$$=\left(\frac{20\times 21}{2}\right)^2-\left(\frac{15\times 16}{2}\right)^2=210^2-120^2=\boxed{29700}.$$

21. What is the product of the greatest common divisor and least common multiple of 20 and 16?

    Answer: 320

    Solution 1: Note that the least common multiple of 80 and the greatest common divisor is 4. Therefore, $\mathrm{lcm}(20,16)\times\gcd(20,16)=80\times 4=\boxed{320}$.

    Solution 2: The product of the least common multiple and the greatest common factor of any two numbers is just the product of the two numbers. Therefore, $\mathrm{lcm}(20,16)\times\gcd(20,16)=20\times 16=\boxed{320}$.

22. Let S be the set of numbers obtained by permuting the digits of 123. What is the sum of the numbers in S?

    Answer: 1332

    Solution: We note there are 6 permutations of 123. Each digit 1, 2, 3 appears exactly twice in each place. (For example, two of the numbers in S have 1 in the hundreds place, two have 2 in the hundreds place, and two have 3 in the hundreds place). Therefore, our answer is $2 \cdot 111 \cdot (1+2+3) = \boxed{1332}$.

23. Lucas is 20 years old and Raymond is 16 years old. How many years ago was it when Lucas was twice as old as Raymond?

    Answer: 12

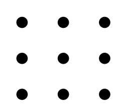
    Solution: Lucas is 4 years older than Raymond. Thus, when Lucas was 8, Raymond was 4. This would have been $\boxed{12}$ years ago.

24. How many distinct positive prime factors does 15015 have?

    Answer: 5

    Solution: Note that $15015 = 15 \times 1001 = 3 \times 5 \times 7 \times 11 \times 13$. Therefore there are $\boxed{5}$ distinct prime factors.

25. How many (nondegenerate) triangles can be made using points from the grid below?



    Answer: 76

    Solution: There are $\binom{9}{3} = 84$ ways to choose three points, but some of these are degenerate. In particular, there are 6 degenerate triangles which lie entirely on a vertical or horizontal line and there are 2 degenerate triangles on the diagonals. Therefore there are $84-8 = \boxed{76}$ nondegenerate triangles.

26. Suppose that you made a pie for Thanksgiving. Lynda ate $\frac{1}{2}$ of the pie. Then Kenia ate $\frac{1}{3}$ of what was left. What fraction of the pie do you have left to eat?

    Answer: $\frac{1}{3}$

    Solution: Start out with 1 pie. After Lynda, there is $1 - \frac{1}{2} = \frac{1}{2}$ of the pie remaining. Then Kenia eats $\frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$ of the pie, so what remains is $\frac{1}{2} - \frac{1}{6} = \boxed{\frac{1}{3}}$ of the pie.

27. Find the area of the circle that circumscribes a triangle with side lengths 6, 8, and 10.

    Answer: 25π

    Solution: We note that the hypotenuse is the diameter of the circle, so the circle has diameter 10, and thus its area is $\boxed{25\pi}$.

28. Alvin the Anteater lives on the number line and initially starts at 0. Every second, he moves one unit in either direction at random. What is the probability that after 13 moves, his location is positive?

Answer: $\frac{1}{2}$

Solution: After an odd number of moves, Alvin cannot be at the origin. For every sequence of 13 moves that ends with Alvin to the right of the origin, the sequence with every move inverted will end up with Alvin on the left, and vice versa. Since each sequence is equally likely, this symmetry means it is equally likely for Alvin to be on either side of the origin, so our answer is $\boxed{\frac{1}{2}}$.

29. What is the area of the largest circle that will fit inside a square with side length 12?

Answer: 36π

Solution: Note that the diameter of this circle is 12, so the radius of the circle is 6. Since the area of a circle is πr^2, the area is $6^2\pi = \boxed{36\pi}$.

30. Anna is a cow tied to the corner of a barn with a rope of length 2. The barn is rectangular with width 1 and length 3. What is the total area of the region over which Anna can roam if she cannot go into the barn?

Answer: $\frac{13\pi}{4}$

Solution: The total area can be split into two parts: one that is a circular sector of radius 2 with area $\frac{3}{4} \cdot 4\pi = 3\pi$, and the other that is a quarter-circle of radius 1, with area $\frac{\pi}{4}$. Thus, Anna can roam a total area equal to the sum of the two areas.

31. Alice only likes integers that are perfect squares. Bob only likes integers that leave a remainder of 1 when divided by 3. How many integers less than 1000 are liked by both Alice and Bob?

Answer: 21

Solution: We note that if $x \equiv 1 \pmod 3$ or $x \equiv 2 \pmod 3$, then $x^2 \equiv 1 \pmod 3$. So for all x that aren't divisible by 3, x^2 leaves a remainder of 1 when divided by 3. 31 is the highest number whose square is less than 1000, so we simply count how many positive numbers up to 31 are not divisible by 3. There are $31 - 10 = \boxed{21}$ such numbers.

32. If $x + y = 20$ and $xy = 16$, what is $x^2 + y^2$?

Answer: 368

Solution: Note that $x^2 + y^2 = (x^2 + 2xy + y^2) - 2xy = (x + y)^2 - 2xy = 20^2 - 2 \cdot 16 = \boxed{368}$.

33. What is the remainder when 20^{16} is divided by 7?

Answer: 1

Solution: The key to this problem is multiplying the factors of 20 one by one, while only considering the remainder. 20 divided by 7 leaves a remainder of 6. Then, $6 \times 20 = 120$ has a remainder of 1 when divided by 7. Notice that since the multiplication would have started with $20^0 = 1$, this is where we started, so the remainders will alternate to be 6 on odd powers of 20, and 1 on even powers. Since 16 is an even power, the remainder will be $\boxed{1}$.

34. 8 students are sitting around a circular table. If you randomly pick 3 of these students, what is the probability they are all seated next to each other?

Answer: $\frac{1}{7}$

Solution: There are 8 different ways to have picked a group of 3 people sitting next to each other, since this is equivalent to simply picking a starting point for the group and adding the two people that are clockwise to that starting point. There are $\binom{8}{3}$ ways to select a group of 3 people, so our answer is $\frac{8}{\binom{8}{3}} = \frac{3!5!}{7!} = \boxed{\frac{1}{7}}$.

35. Suppose that (x, y) is a solution to the equation $x^2 + y^2 = 1$. What is the maximum value of xy?

Answer: $\mathbf{\frac{1}{2}}$

Solution: Intuitively, it would seem the maximum would be $\frac{1}{2}$ when $x = y = \pm\frac{1}{\sqrt{2}}$. We then need to prove this is indeed the maximum. If $x = 0$, then $y = 1$, and $xy = 0$. Otherwise, we can let $y = ax$ for some a, and maximize ax^2 subject to $(1 + a^2)x^2 = 1$. This gives $x^2 = \frac{1}{1+a^2}$, so we just wish to maximize $\frac{a}{1+a^2}$. Assume that $\frac{a}{1+a^2} > \frac{1}{2}$. We then see that $2a > 1 + a^2$, so $0 > a^2 + 2a + 1 = (a + 1)^2$. Since $(a + 1)^2$ is always nonnegative, that inequality cannot be true, so our maximum can be no greater than $\boxed{\frac{1}{2}}$.

36. A teacher is returning exams to 5 students. In how many ways can she return the exams such that no student gets their own exam?

Answer: 44

Solution: One solution is to use the Principle of Inclusion-Exclusion and compute

$$\sum_{i=0}^{5} (-1)^i \binom{5}{i} (5 - i)!$$

For another solution, let $f(x)$ be the number of ways to solve our problem for x people. Clearly, $f(1) = 0$ and $f(2) = 1$, and we can try to find a recursive solution. Let y be the student who receives the first student's paper. We then have two cases.

If the first student receives y's paper, then they have simply swapped papers, and we just need to solve the same problem for the remaining $x - 2$ people.

On the other hand, if the first person did not receive $y's$ paper, then we need to assign papers for every person except y with the same constraint as before, and that the first person cannot receive y's paper. Thus all $x - 1$ people we still need to assign papers to have exactly one paper that they cannot get back.

There are $x - 1$ choices for person y, so our recurrence is thus $f(x) = (x - 1)(f(x - 2) + f(x - 1))$. We can then compute $f(3) = 2(1 + 0) = 2$, $f(4) = 3(2 + 1) = 9$, and $f(5) = 4(9 + 2) = \boxed{44}$.

37. Freddy rolls a 20-sided die (with sides labeled 1 through 20) 16 times, and sums the results. What is the most probable sum?

Answer: 168

Solution: One can easily notice the pattern that when n dice are rolled and n is even, the most probable sum is $10.5n$. In this case, we have $10.5 \cdot 16 = \boxed{168}$.

38. What is the coefficent of the x^2 term in the polynomial $(2x + 4)(3x^2 + 4)$?

Answer: 12

Solution: Expand the polynomial as $(2x + 4)(3x^2 + 4) = 6x^3 + 12x^2 + 8x + 16$. Therefore, the coefficent of the x^2 term is $\boxed{12}$.

39. How many real solutions are there for the equation $\frac{x}{y} + \frac{y}{x} = 1$?

Answer: 0

Solution: We know that $x, y \neq 0$. We consider the case where x and y are the same sign. In this case, both $\frac{x}{y}$ and $\frac{y}{x}$ are positive, and at least one of them has to be greater than or equal to 1, so their sum must also be greater than 1. If they are of different signs, then both of our terms are negative, so they cannot sum to 1. Therefore, we conclude that there are no real solutions.

40. How many three digit numbers have digits whose product is 12?

Answer: 15

Solution: When we arrange the digits from least to greatest, we see that the possible digits are 126, 134 and 223. Both 126 and 134 can be arranged in $3! = 6$ ways, while 223 can be arranged in $\frac{3!}{2!} = 3$ ways. So there are $6 + 6 + 3 = \boxed{15}$ such three digit numbers.

41. Define the operation $a \otimes b = a^2 + b^2$. Find $(1 \otimes 2) \otimes 3$.

Answer: 34

Solution: We compute $1 \otimes 2 = 1^2 + 2^2 = 5$, so therefore $(1 \otimes 2) \otimes 3 = 5 \otimes 3 = 5^2 + 3^2 = \boxed{34}$.

42. Let C be a circle of radius 21 with center O, and let A be a point 29 units away from O. Let B be a point on C such that $\overrightarrow{AB}$ is tangent to C. Compute the area of triangle ABO.

Answer: 210

Solution: We note that since $\overrightarrow{AB}$ is tangent to C, ABO must be a right triangle with legs $\overline{BO}$ and $\overline{AB}$. $\overline{BO}$ is a radius, so it has length 21. We can then use the Pythagorean Theorem to compute the length of $\overline{AB}$. This is

$$\overline{AB}^2 = \overline{AO}^2 - \overline{BO}^2 = 29^2 - 21^2 = 400.$$

We then have $\overline{AB} = 20$, so the area of ABO is $\frac{20 \cdot 21}{2} = \boxed{210}$.

43. Consider the line segment from $(0, 0)$ to $(320, 180)$. How many points on this line segment have coordinates that are both integers? Include the endpoints $(0, 0)$ and $(320, 180)$.

Answer: 21

Solution: This line has a slope of $\frac{180}{320} = \frac{9}{16}$. So the equation of the line is $y = \frac{9}{16}x$. In order for y to be an integer, x must be divisible by 16. The values that x can take on are $0, 16, 32, \ldots, 320$. There are $\boxed{21}$ numbers in this list.

44. Define $a \oplus b = a + ab$. Compute $(7 \oplus 8) \oplus 9 - 7 \oplus (8 \oplus 9)$.

Answer: 63

Solution 1: Compute $7 \oplus 8 = 63$ and $63 \oplus 9 = 630$. Then compute $8 \oplus 9 = 80$ and $7 \oplus 80 = 567$. Now conclude that $630 - 567 = \boxed{63}$.

Solution 2: Note that $a \oplus b = a(b + 1)$. So $(a \oplus b) \oplus c = a(b + 1)(c + 1)$. On the other hand, $a \oplus (b \oplus c) = a(b(c + 1) + 1)$. Therefore

$$\begin{aligned}(a \oplus b) \oplus c - a \oplus (b \oplus c) &= a(b + 1)(c + 1) - a(b(c + 1) + 1) \\ &= a((b + 1)(c + 1) - (b(c + 1) + 1)) \\ &= a(bc + b + c + 1 - bc - b - 1) \\ &= ac.\end{aligned}$$

Taking $a = 7$ and $c = 9$, we see that $7 \cdot 9 = \boxed{63}$.

45. How many ways are there to choose two different students out of a classroom of 64 students?

Answer: 2016

Solution: There are $\binom{64}{2} = \boxed{2016}$ ways to choose the students.

46. If a rectangle has perimeter 2016, what is its maximum possible area?

Answer: 254016

Solution: The area of the rectangle can be maximized by making it a square. The side length of the square would be $\frac{2016}{4} = 504$. The area is $504^2 = \boxed{254016}$.

47. How many proper divisors does 2016 have?

Answer: 35

Solution: The prime factorization of 2016 is $2016 = 2^5 \cdot 3^2 \cdot 7$. Every divisor is of the form $2^a \cdot 3^b \cdot 7^c$, where $a \leq 5$, $b \leq 2$, and $c \leq 1$, so there are $6 \cdot 3 \cdot 2 = 36$ choices. However, since we want only proper divisors, 2016 itself does not count, so our answer is $36 - 1 = \boxed{35}$.

48. How many positive integers less than or equal to 2016 are multiples of 20 or 16, but not both?

Answer: 176

Solution: There are $\lfloor \frac{2016}{20} \rfloor = 100$ multiples of 20 and $\lfloor \frac{2016}{16} \rfloor = 126$ multiples of 16. However, we are overcounting the multiples of 80, which are both 20 and 16, when we shouldn't be counting them at all. There are $\lfloor \frac{2016}{80} \rfloor = 25$ multiples of 80. Thus, our answer is $100+126-2\times25 = \boxed{176}$.

49. Find the 2016th letter of the infinite sequence: $BERKELEYBERKELEY\ldots$.

Answer: *Y*

Solution: The sequence repeats every eight letters. $2016 \equiv 8 \pmod 8$, so the 2016th letter is the same as the 8th letter, which is $\boxed{Y}$.

50. What year is it?

Answer: 2016

Solution: As evidenced by the choice of numbers for many of the questions, the year is likely $\boxed{2016}$.